UNIVERSITAT POLITÈCNICA DE CATALUNYA
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ETSEIB

Mètodes Numèrics:

A First Course on Finite Elements

Dept. Matemàtiques
ETSEIB - UPC BarcelonaTech

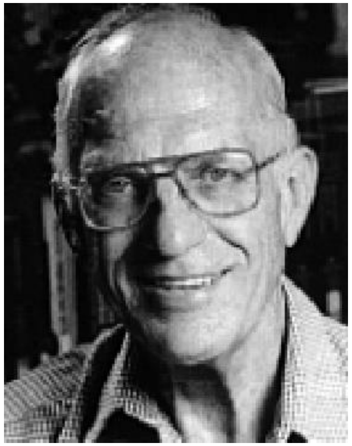
Short History: annals

- 1943: **Richard Courant**, a mathematician, described a piecewise polynomial solution for the torsion problem of a shaft of arbitrary cross section. Even with holes. The early ideas of FEM date back to a 1922 book by Hurwitz and Courant.

1888-1972: born in Lublintz Germany
Student of Hilbert and Minkowski in Gottingen Germany
Ph.D in 1910 under Hilbert's supervision.
1934: moved to New York University, founded the Courant Institute

Short History: name

- 1960: The name "**finite element**" was coined by structural engineer **Ray Clough** of the University of California**.**

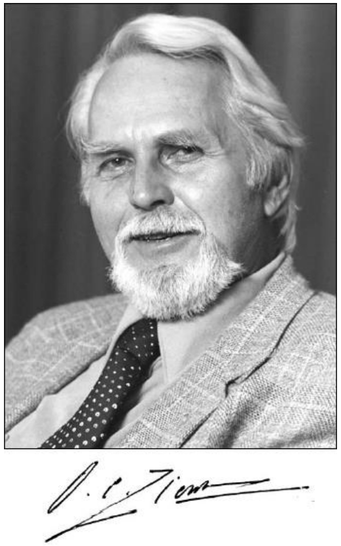


1920: born in Seattle
Professor emeritus of Structural Engineering at UC Berkley
Studied for Boeing the vibration of wing airplanes
Ph.D from MIT.
Well known as a earthquake engineer

Short History: divulgation

- 1967: **Olgierd C. Zienkiewicz** published **The first book** on the FEM

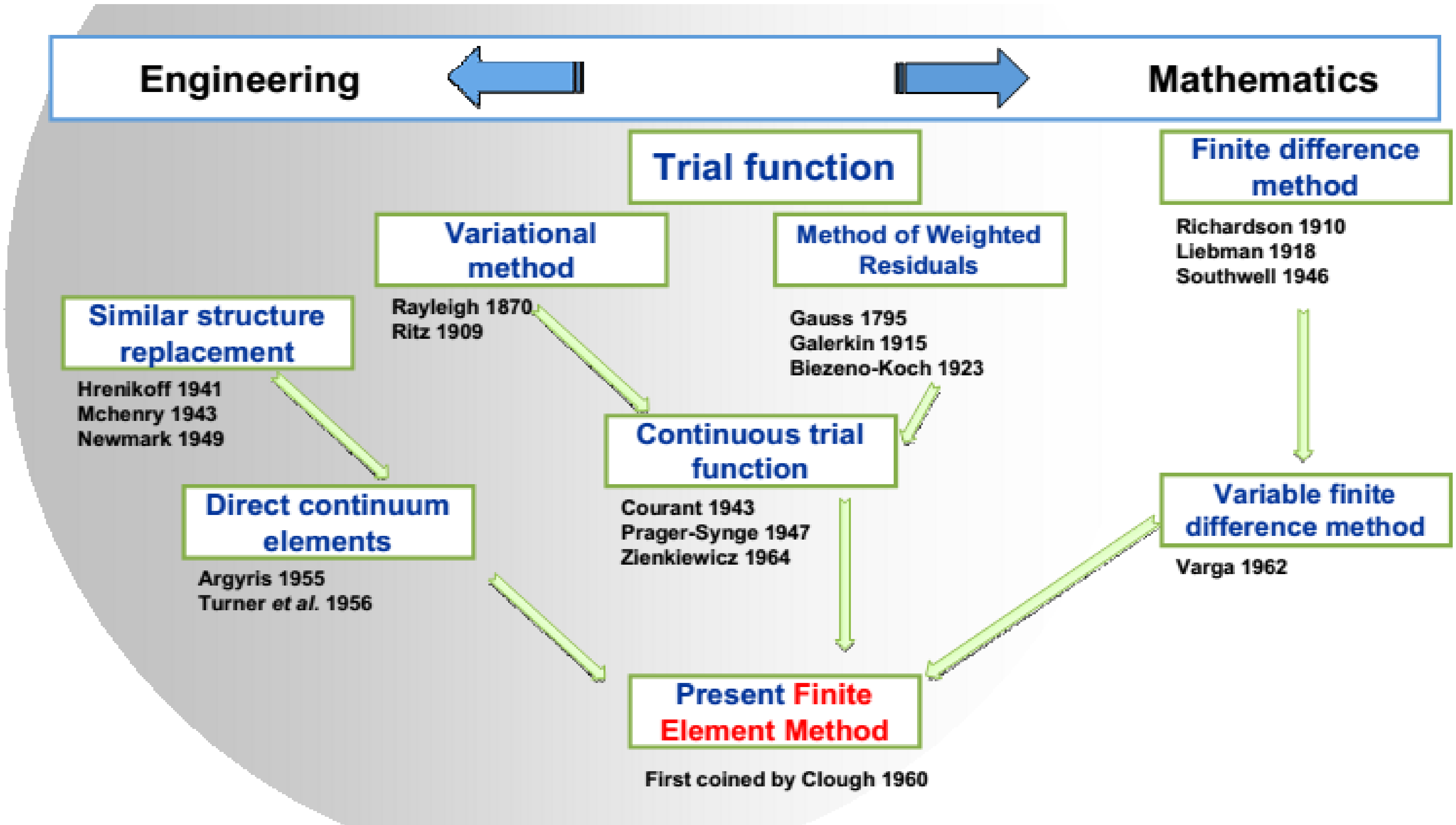
O.C. ZIENKIEWICZ (with Y.K. CHEUNG), (1967) The Finite Element Method in Continuum and Structural Mechanics , McGraw Hill, 272 pp (translated into Japanese and Russian)



1921-2009: born in Caterham, England.
Professor Department of Civil Engineering at Swansea University
Ph.D from Imperial College London.
He published nearly 600 papers and wrote or edited more than 25 books

For more information:
https://en.wikipedia.org/wiki/Finite_element_method

Short History: FEM-Mathematics



The Finite Elements Approach

Following:

http://www.colorado.edu/engineering/cas/courses.d/IFEM.d/

Dept. Matemàtiques

General Idea:

Divide and Conquer!!

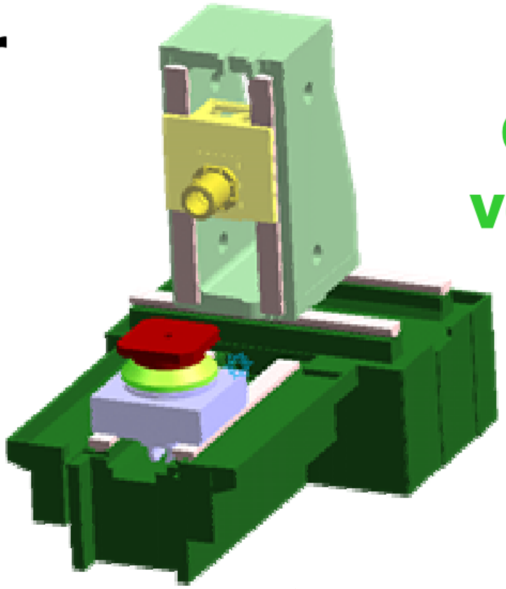
General Idea: **Divide and Conquer**

Vertical machining center

Elastic deformation

Thermal behavior

etc.



Geometry is very complex!

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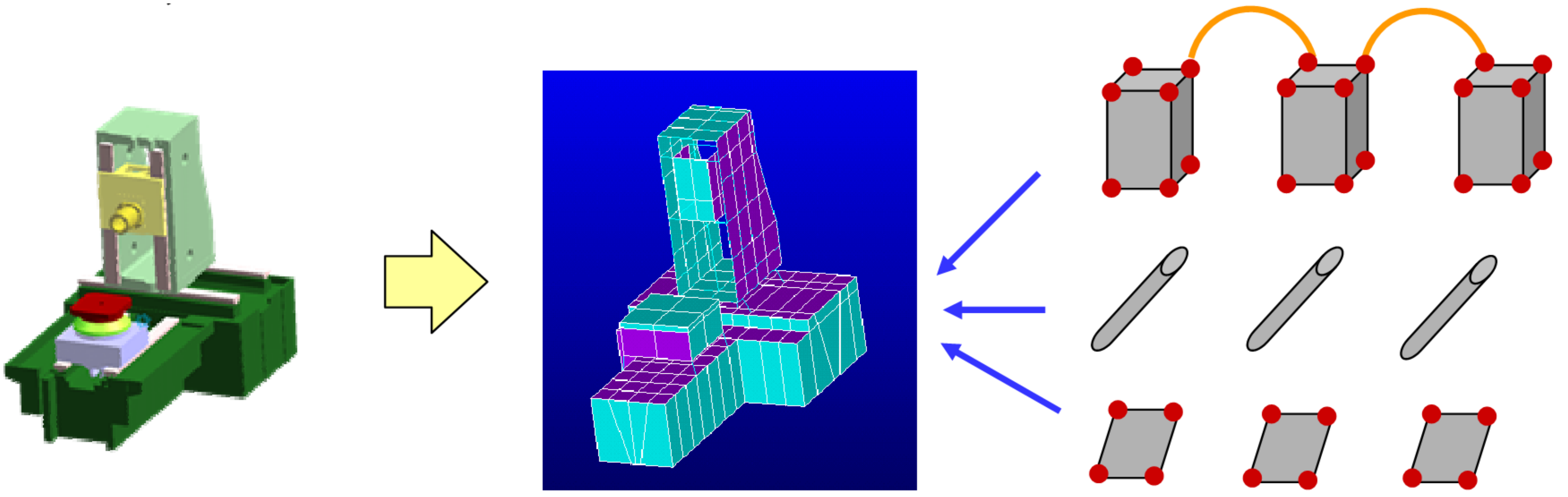
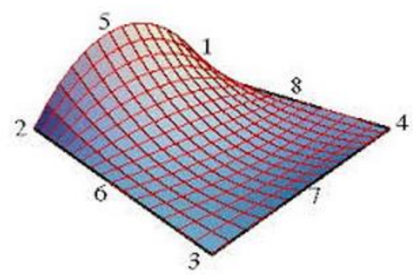
General Idea: **Divide and Conquer**

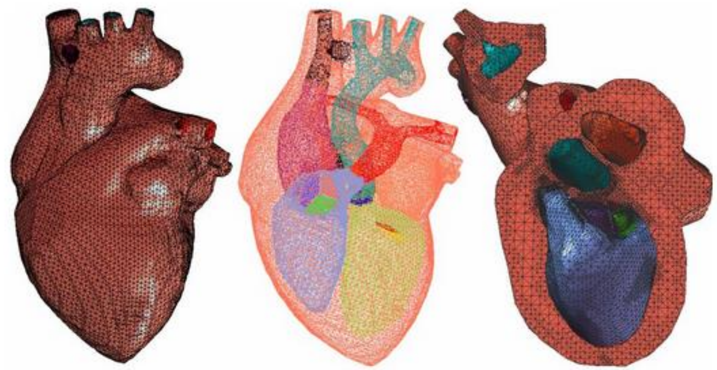
- Subdivide the domain in different parts:

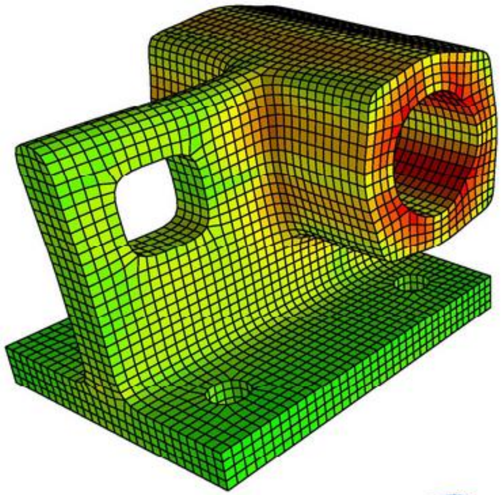


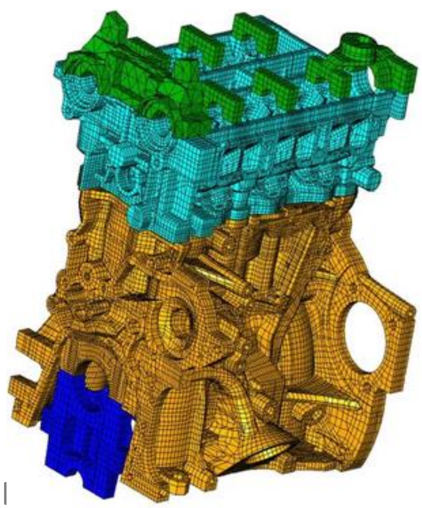
General Idea: **Divide and Conquer**

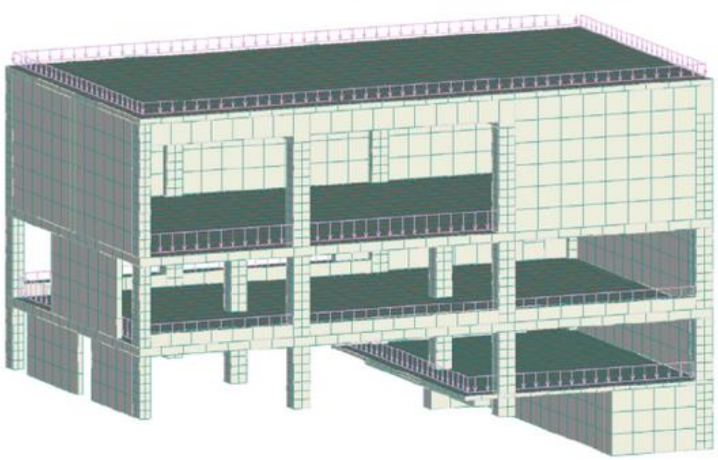
- This can be done in several different fields:

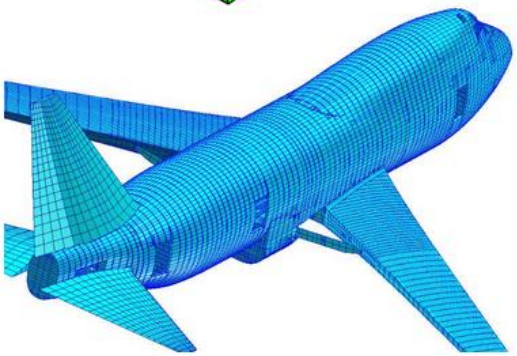




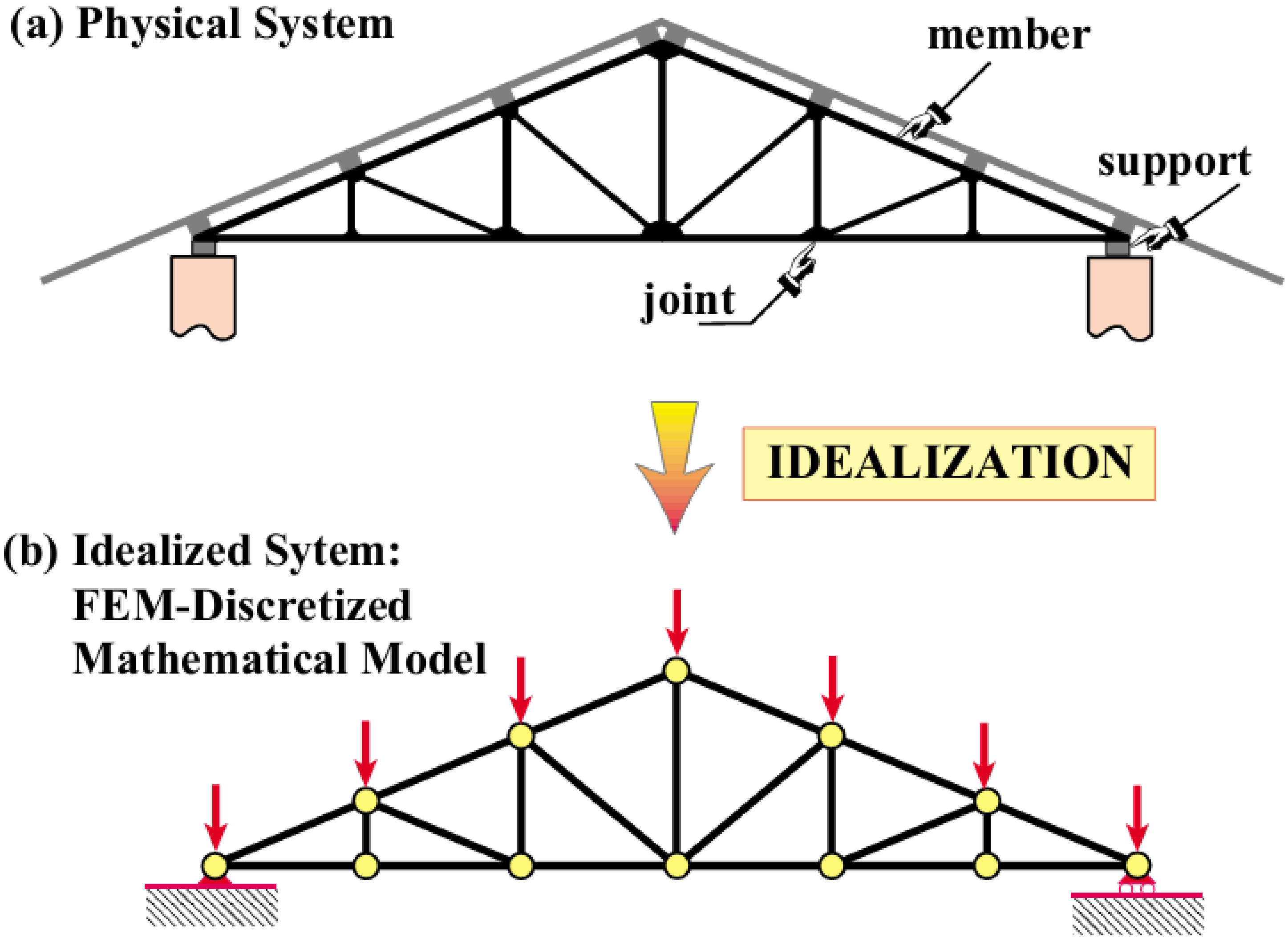




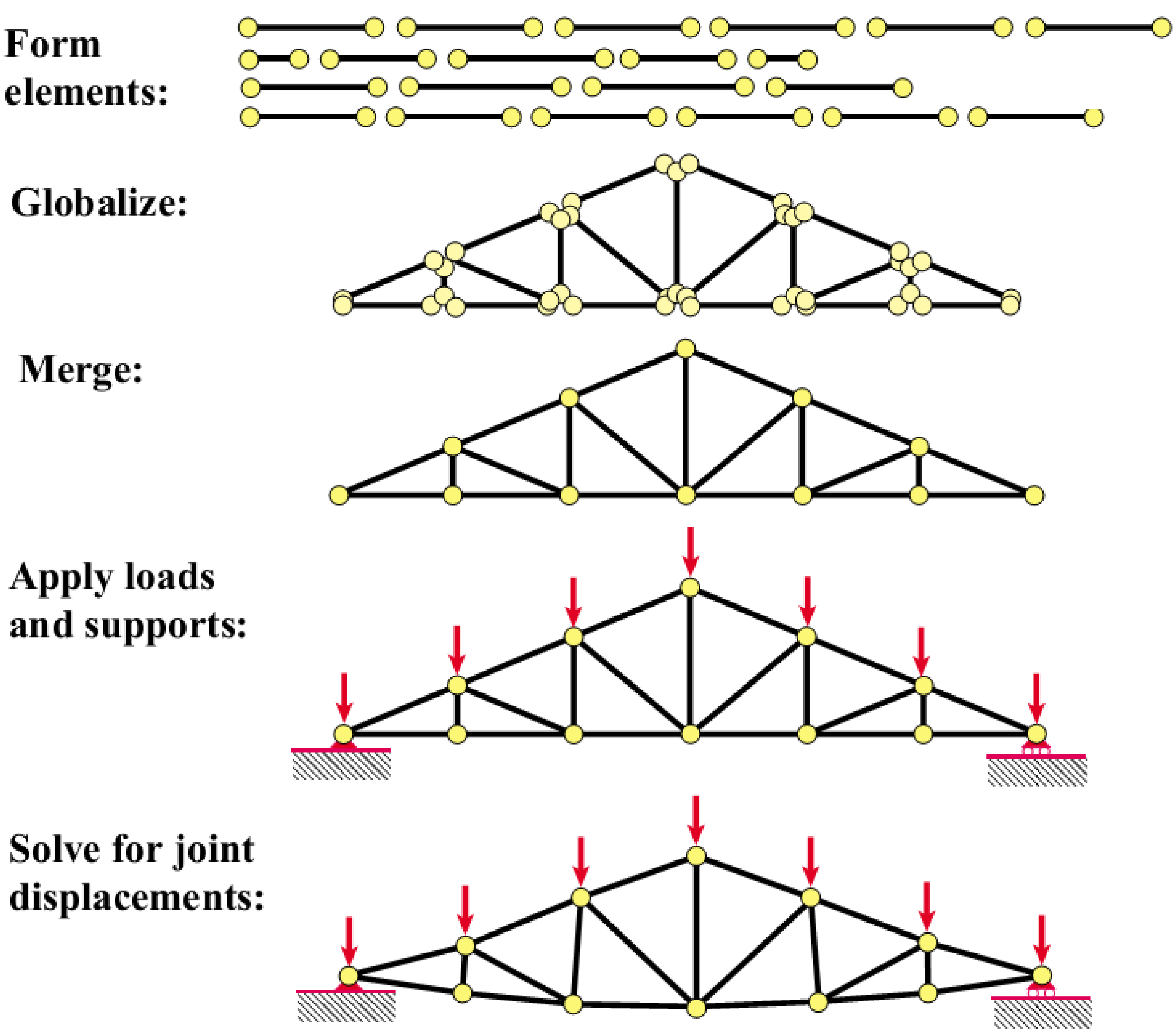




General Idea: **Example**



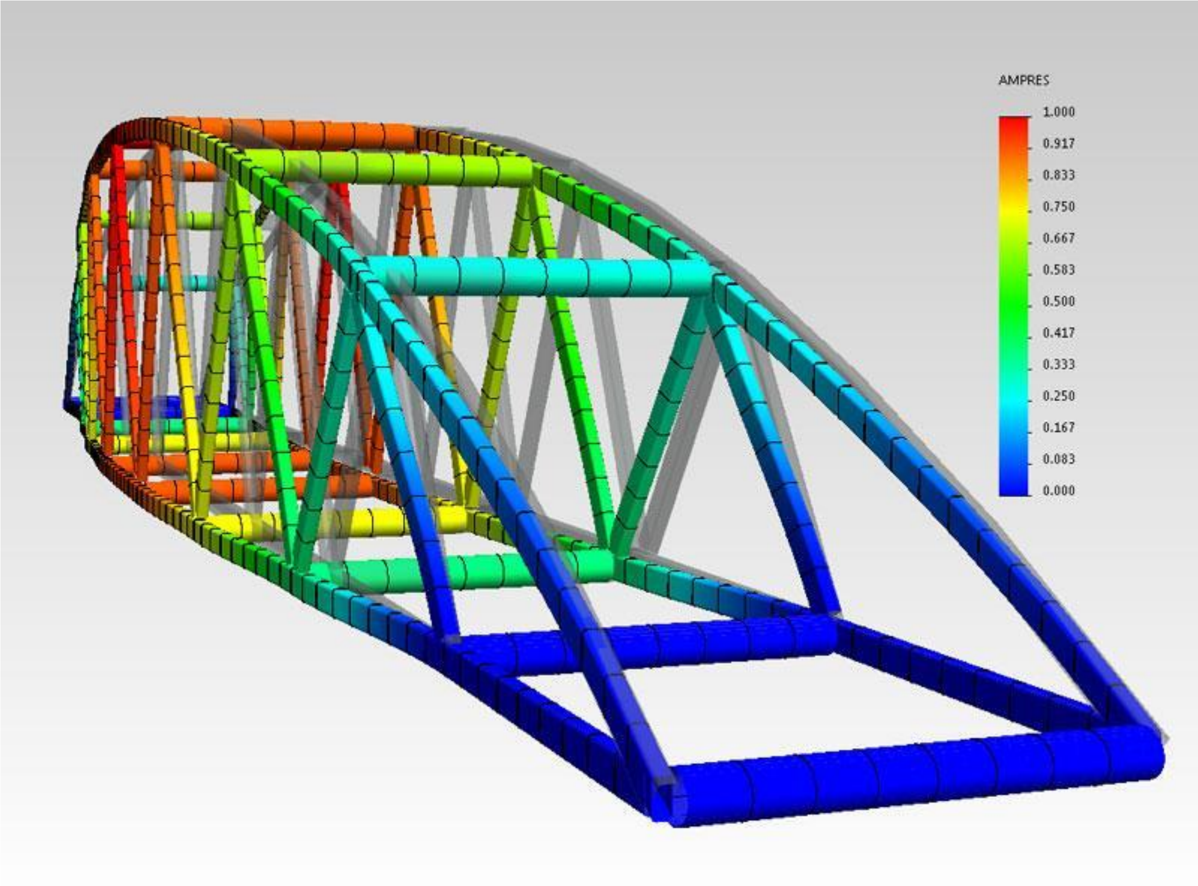
General Idea: **Example**



Generalized Approach

- **Step 1:** State the **physical problem**

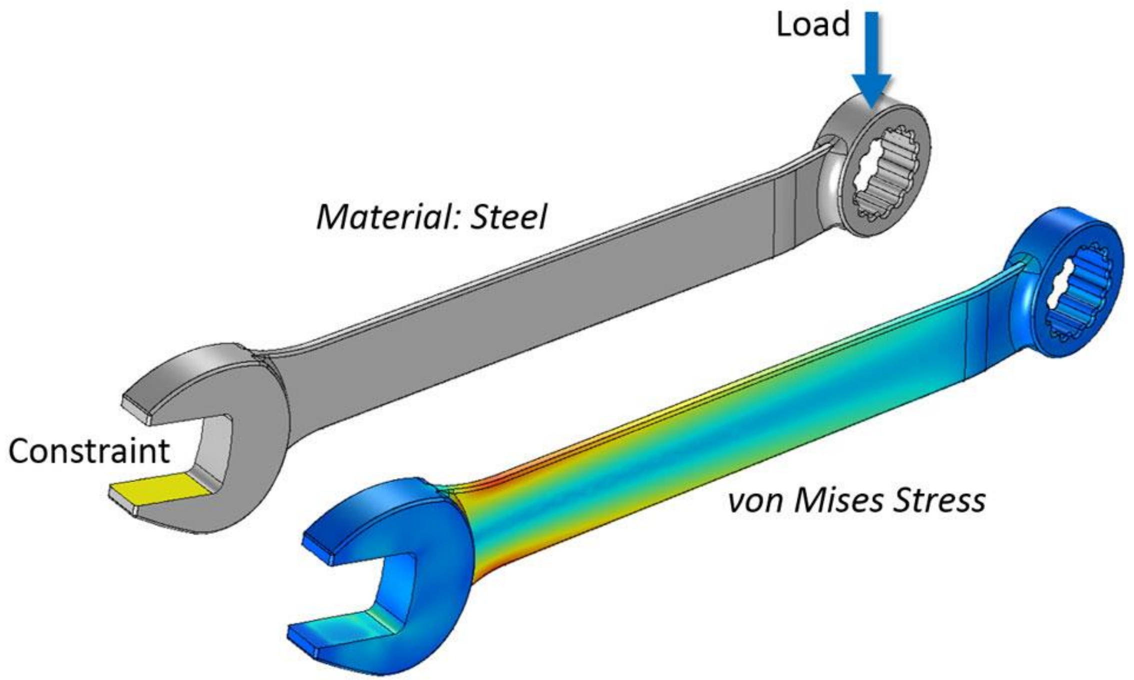
STRUCTURAL



Generalized Approach

- **Step 1:** State the **physical problem**

STRUCTURAL



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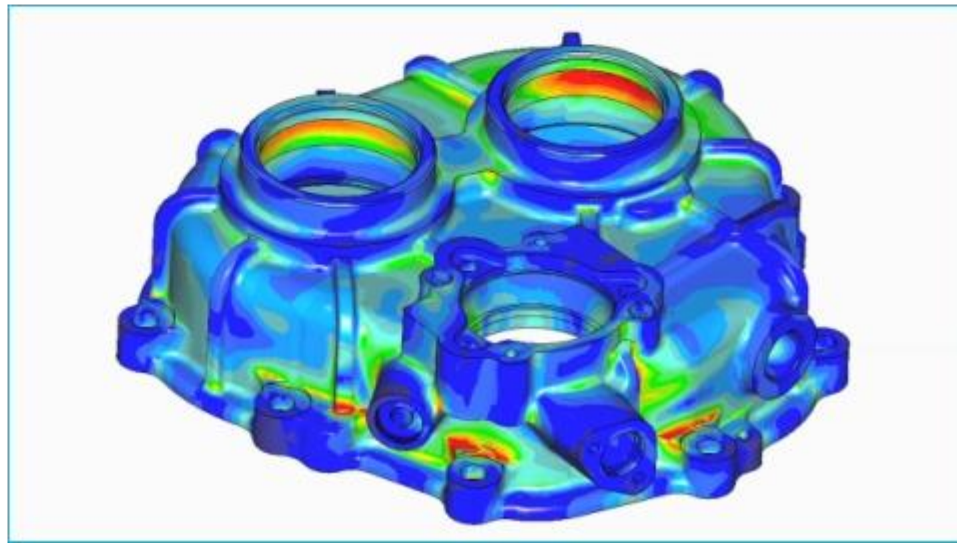
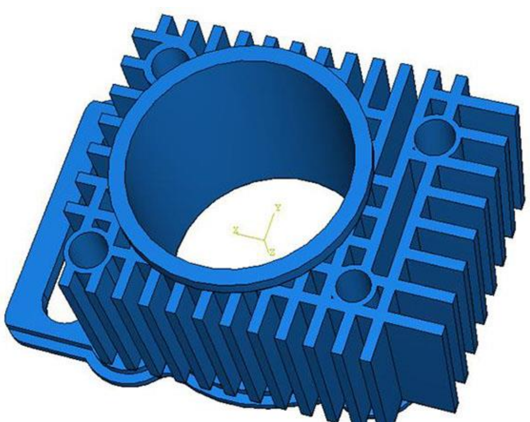
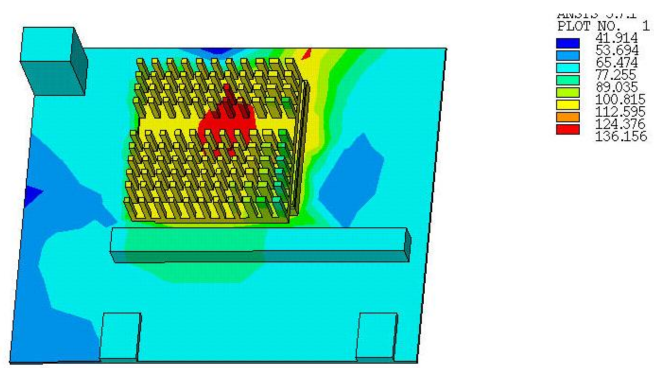
Generalized Approach

- **Step 1:** State the **physical problem**

THERMAL



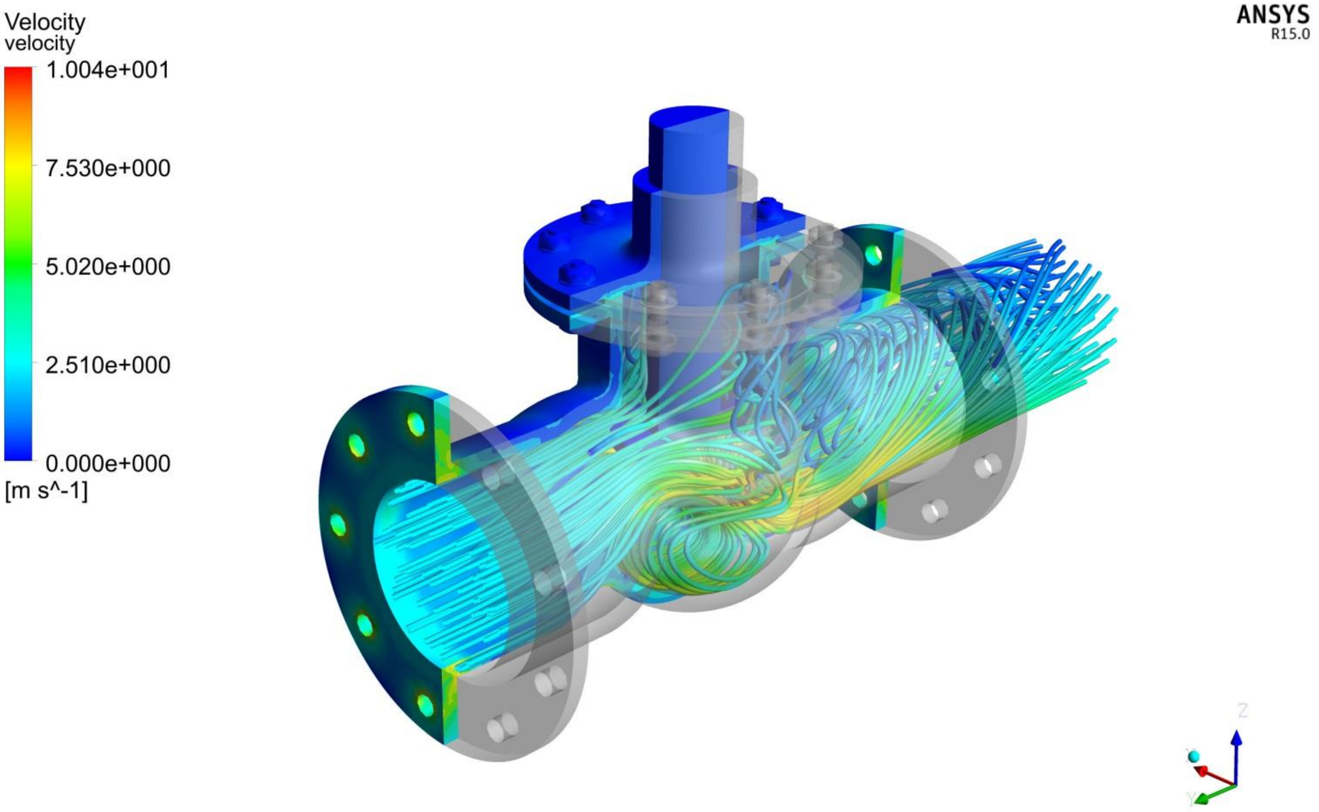




Generalized Approach

- **Step 1:** State the **physical problem**

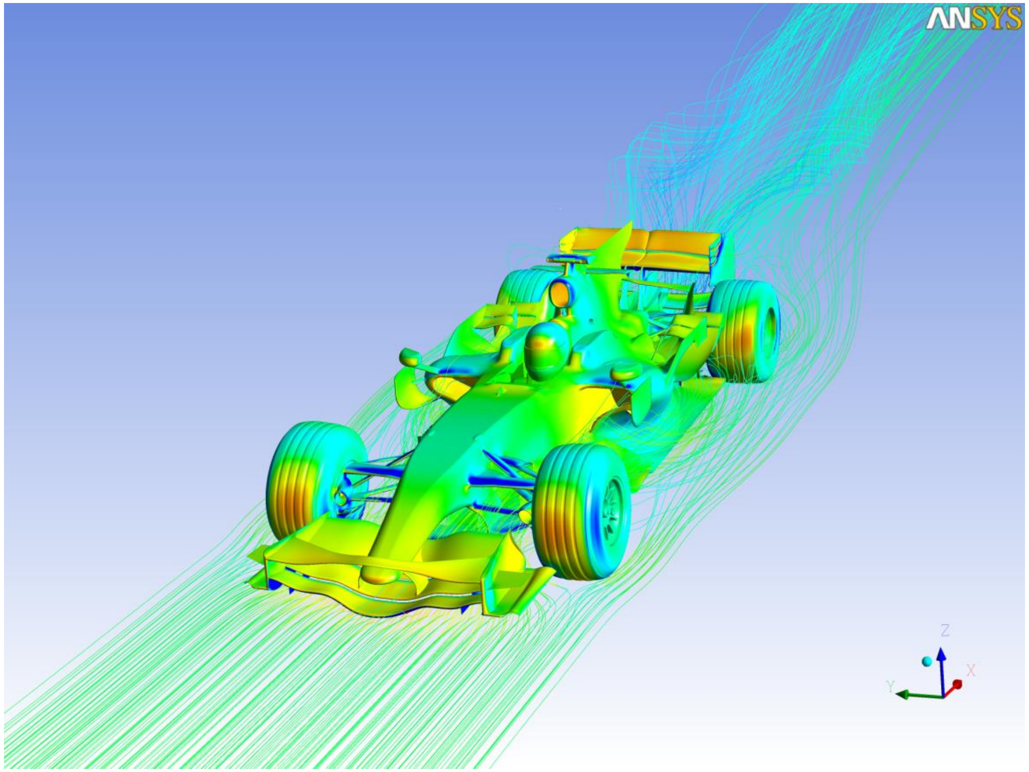
FLUIDS



Generalized Approach

- **Step 1:** State the **physical problem**

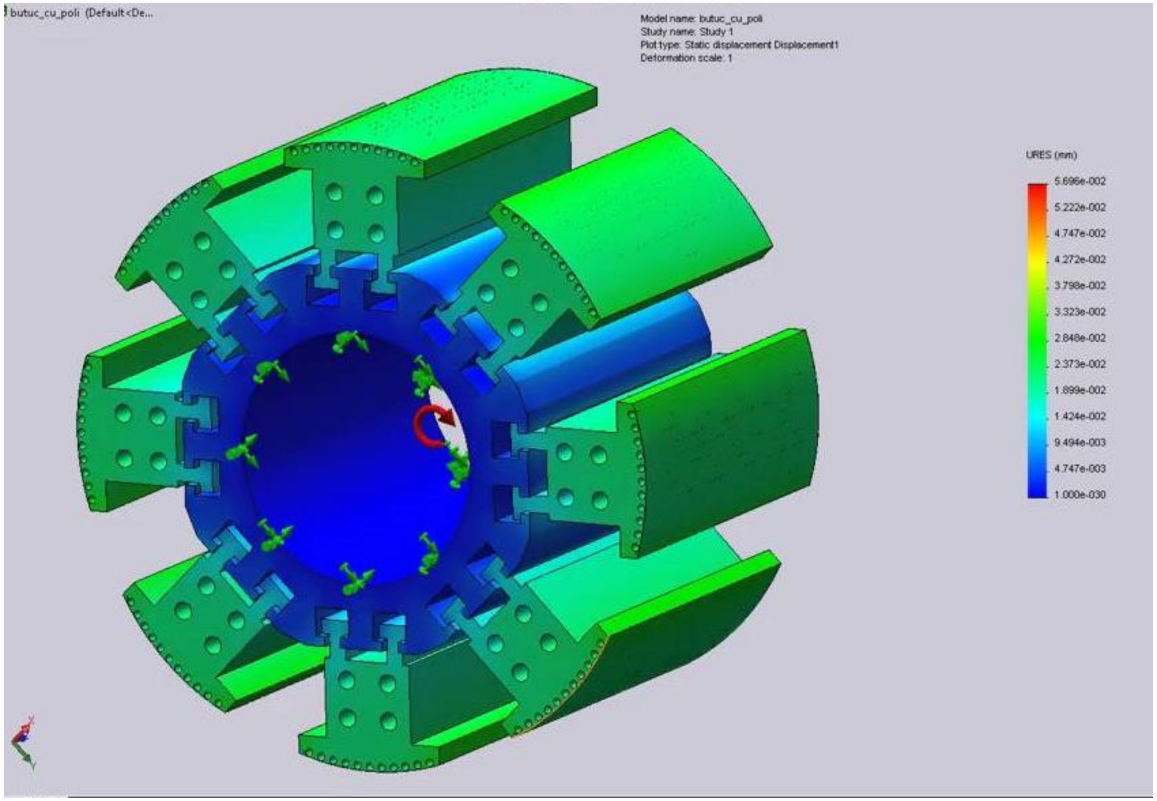
AERODYNAMICS



Generalized Approach

- **Step 1:** State the **physical problem**

Electromagnetism



Generalized Approach

- **Step 1:** State the **physical problem**
- **Step 2:** State the **Mathematical model**

Generalized Approach

- **Step 2:** State the **Mathematical model**

Partial Differential Equations **(PDE)**

Heat Transfer (1-D):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$u = u(t, x)$ temperature
k thermal conductivity coeff.

Analogous notation:

$$u_t = ku_{xx}$$

Generalized Approach

- **Step 2:** State the **Mathematical model**

Partial Differential Equations **(PDE)**

Wave equation (1-D):

$$\frac{\partial u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u = u(t, x)$ high value
c velocity wave propagation

Analogous notation:

$$u_{tt} = c^2 u_{xx}$$

Generalized Approach

- **Step 2:** State the **Mathematical model**

Partial Differential Equations **(PDE)**

Fluid equations (2-D): Navier-Stokes

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)+\rho g_y.$$

(u_x, u_y) fluid velocity components
μ viscosity
ρ density
p pressure

Generalized Approach

- **Step 2:** State the **Mathematical model**

Partial Differential Equations **(PDE)**

Structural Analysis (2-D): Plane Stress

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1+v}{2}\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y}\right)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1+v}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y}\right)$$

(u, v) displacement components
ν poisson ratio

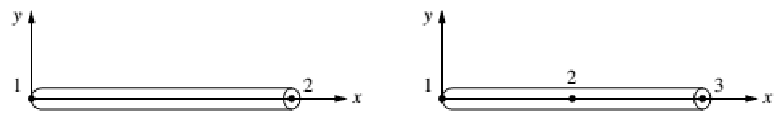
Generalized Approach

- **Step 1:** State the **physical problem**
- **Step 2:** State the **Mathematical model**
- **Step 3: Numerical Solution**

Generalized Approach

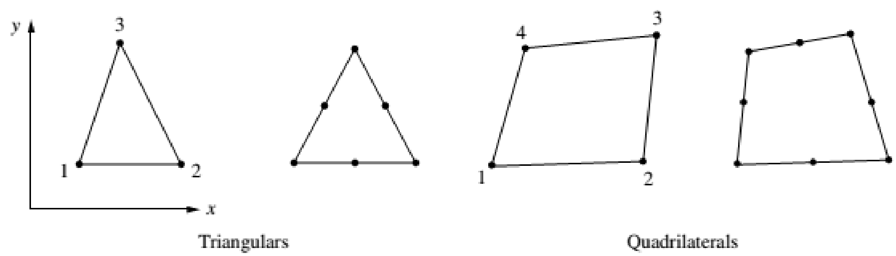
- **Step 3: Numerical Solution**

3.1 Domain discretization: Choose elements



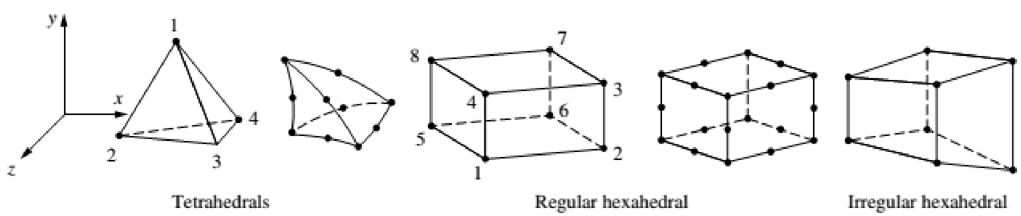
(a) Simple two-noded line element (typically used to represent a bar or beam element) and the higher-order line element

Nodes



Elements

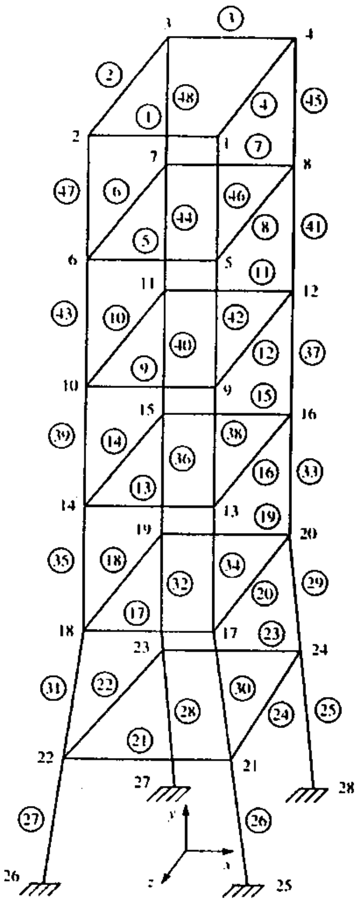
(b) Simple two-dimensional elements with corner nodes (typically used to represent plane stress/strain) and higher-order two-dimensional elements with intermediate nodes along the sides



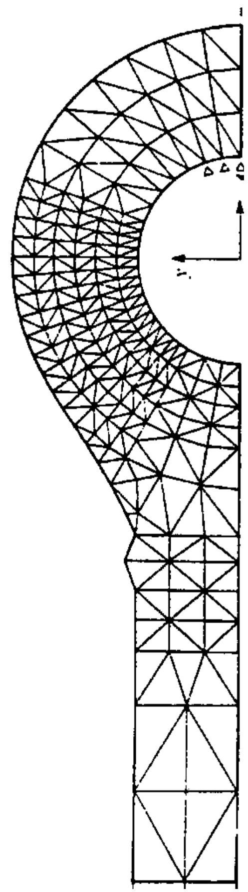
(c) Simple three-dimensional elements (typically used to represent three-dimensional stress state) and higher-order three-dimensional elements with intermediate nodes along edges

Generalized Approach

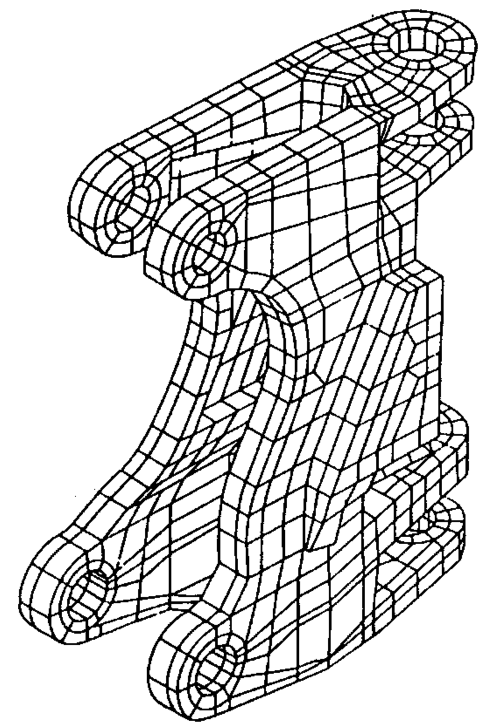
- **Step 3.1 Domain discretization:**



One-Dimensional Frame Elements



Two-Dimensional Triangular Elements



Three-Dimensional Brick Elements

Generalized Approach

- **Step 3.2. Numerical solution (Linear Systems)**
  - For each element we obtain a **small linear system**

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & k_{3n} \\ \vdots & & & & \vdots \\ k_{n1} & & & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{Bmatrix}$$

Generalized Approach

- **Step 3.2. Numerical solution (Linear Systems)**
  - For each element we obtain a **small linear system**

Exemple: Structural 1-dim bar:

$$f_e = K_e u_e$$

u_e: displacements, 2
f_e: forces, 2
K_e: Stress Matrix, 2x2

$$K_e = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

E: **Young Modulus** (elasticity coef.)
A: **section Area** of the bar
L: **Length** of the bar

Generalized Approach

- **Step 3.2. Numerical solution (Linear Systems)**
  - **Assembling** all elements we obtain a **BIG global linear system**

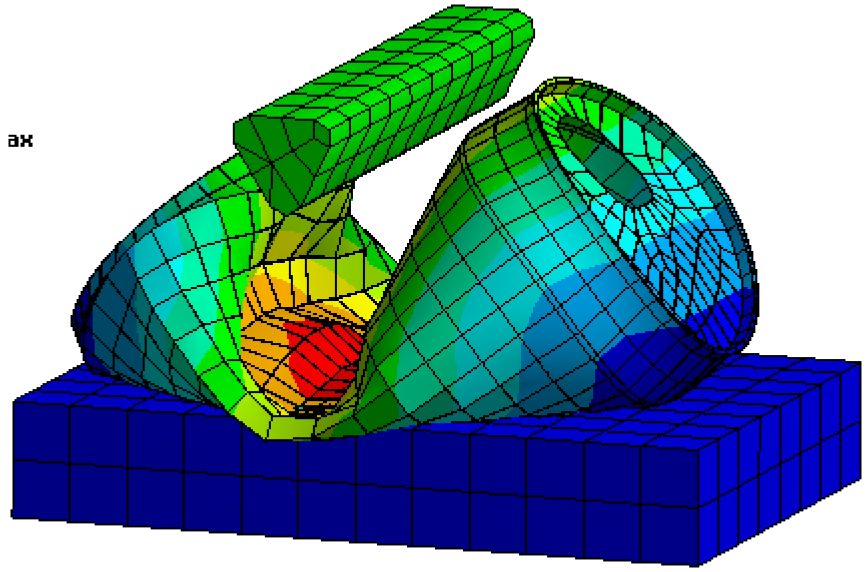
$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & & & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix}$$

Generalized Approach

- **Step 1:** State the **physical problem**
- **Step 2:** State the **Mathematical model**
- **Step 3: Numerical Solution**
- **Step 4: Solution Postprocess**

Generalized Approach

- **Step 4: Solution Postprocess**
  - Compute other magnitudes (Stress, thermal flow, etc)
  - Plot results



Generalized Approach

Summary

Physical Problem
(Modeling Equations)

↓

Domain discretization
(Elements, Nodes)

↓

Problem Solution
(Linear Systems)

↓

Solution Postprocess
(Show new results)

Generalized Approach

Summary

Physical Problem
(Modeling Equations)

↓

Domain discretization
(Elements, Nodes)

↓

Problem Solution
(Linear Systems)

↓

Solution Postprocess
(Show new results)

Mathematical tools

Partial Differential Equations

Weak Problem Formulation

Shape functions (interpolation)

Numerical Integration

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Generalized Approach

Summary

Physical Problem
(Modeling Equations)

↓

Domain discretization
(Elements, Nodes)

↓

Problem Solution
(Linear Systems)

↓

Solution Postprocess
(Show new results)

Mathematical steps

Element dimension and shape

Meshing general domains

Impose **Boundary Conditions**

Obtain element matrices

Enumerate Element and Nodes

Global Assembly

Generalized Approach

Summary

Physical Problem
(Modeling Equations)

↓

Domain discretization
(Elements, Nodes)

↓

Problem Solution
(Linear Systems)

↓

Solution Postprocess
(Show new results)

Mathematical steps

Matrix shape and reordering

Vector and Matrix Manipulation

Numerical methods for Linear Systems

Generalized Approach

Summary

Physical Problem
(Modeling Equations)

↓

Domain discretization
(Elements, Nodes)

↓

Problem Solution
(Linear Systems)

↓

Solution Postprocess
(Show new results)

←

Mathematical steps

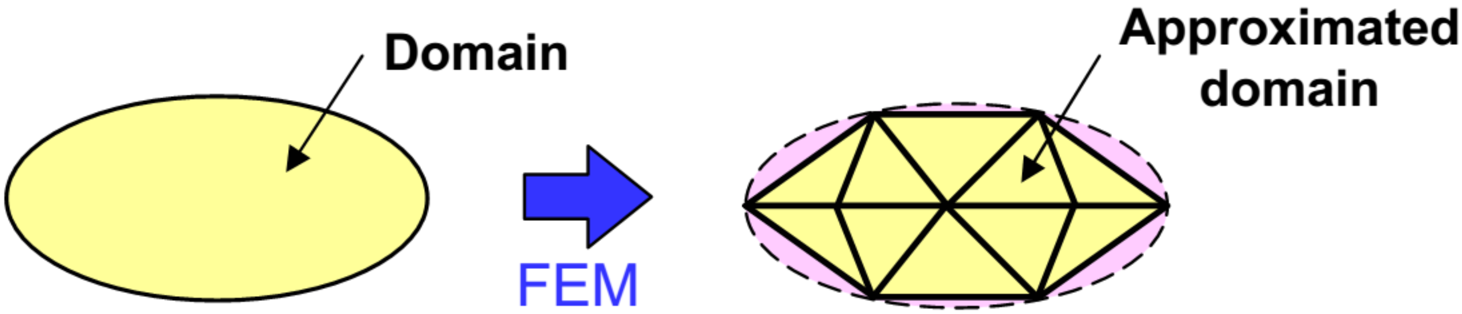
Compute secondary variables

Plot results

Evaluate solution: critical points, remeshing, estimate errors, etc.

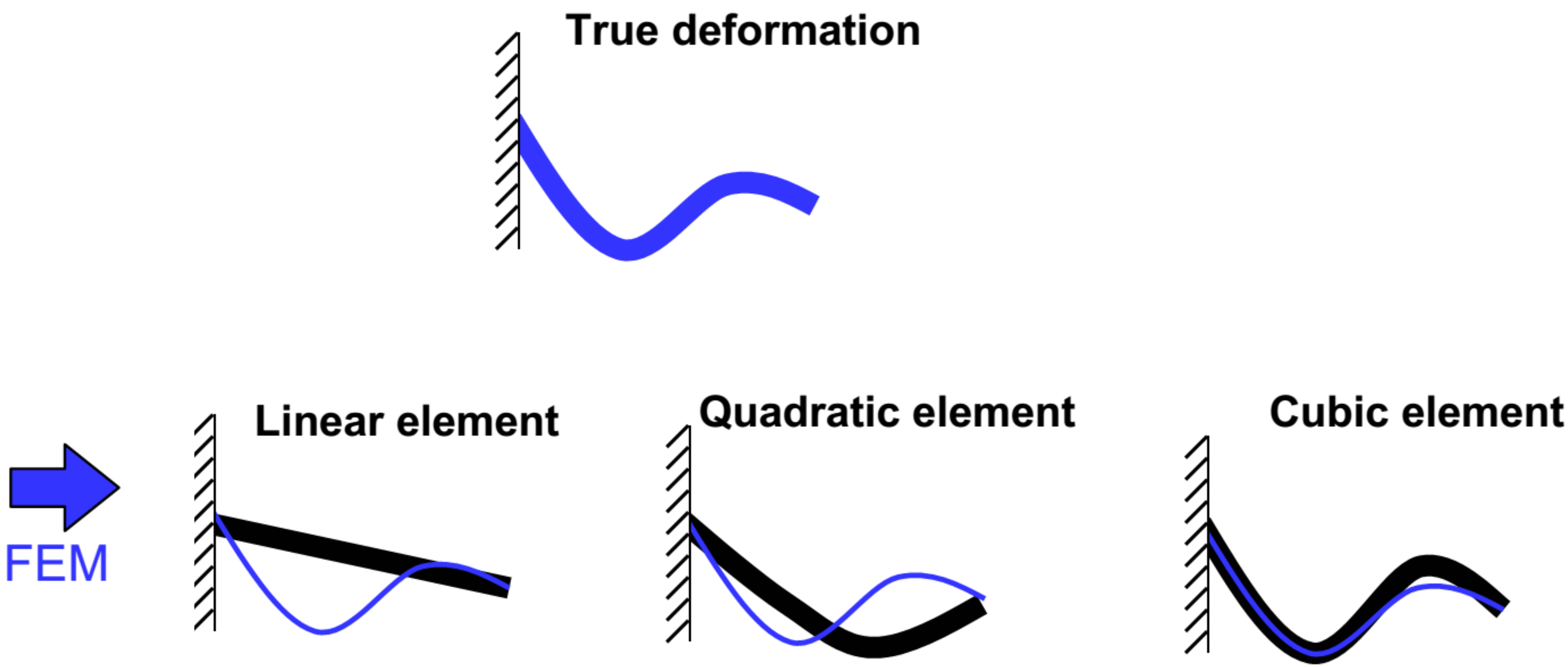
Error Sources

- **Simplified Geometry**



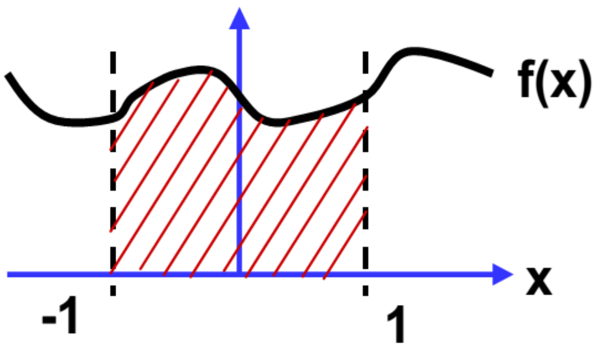
Error Sources

- **Simplified Geometry**
- **Shape function approximation (polynomial)**



Error Sources

- **Simplified Geometry**
- **Shape function approximation (polynomial)**
- **Numerical integration (Gauss Quadrature)**



$$\text{Area:} \quad \int_{-1}^{1} f(x)\,dx \approx f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

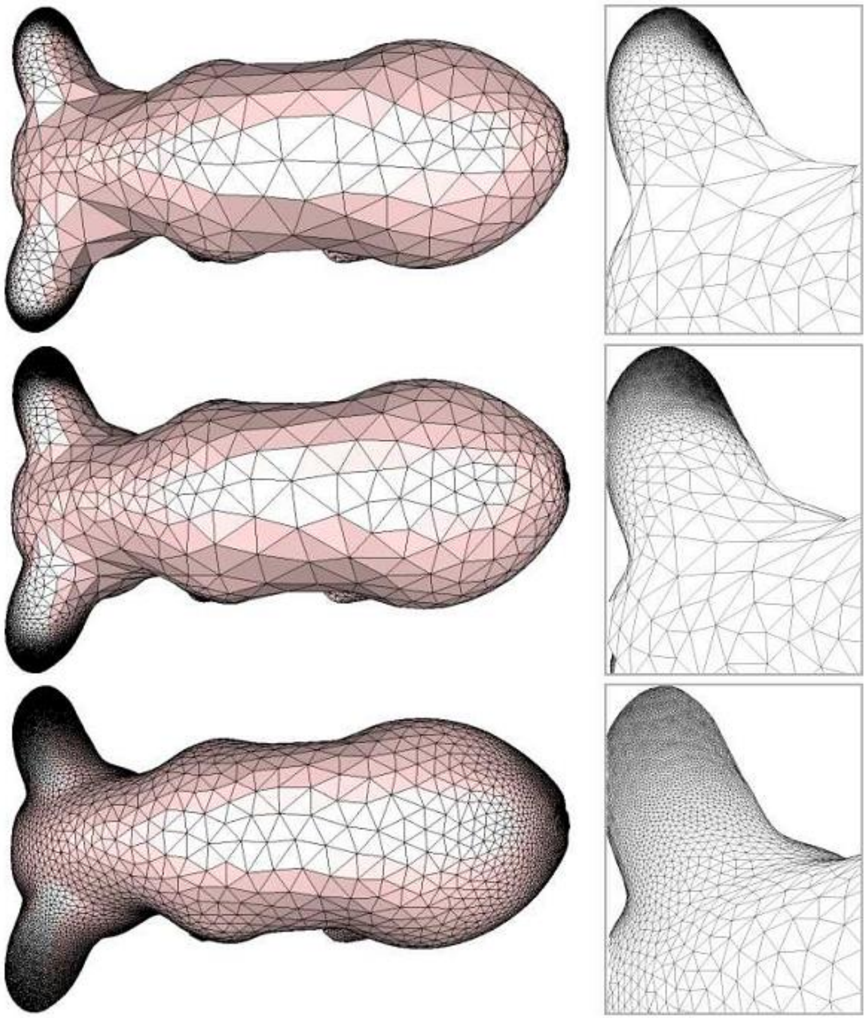
Solution convergence strategies

- **How to minimize the error?**
  - Actual solution is not known
  - Strategy:
    - Compare between **two different** results of the same problem
    - Check how much it differs
    - Take a third result. There is **convergence**?

Solution convergence strategies

- **H-method**

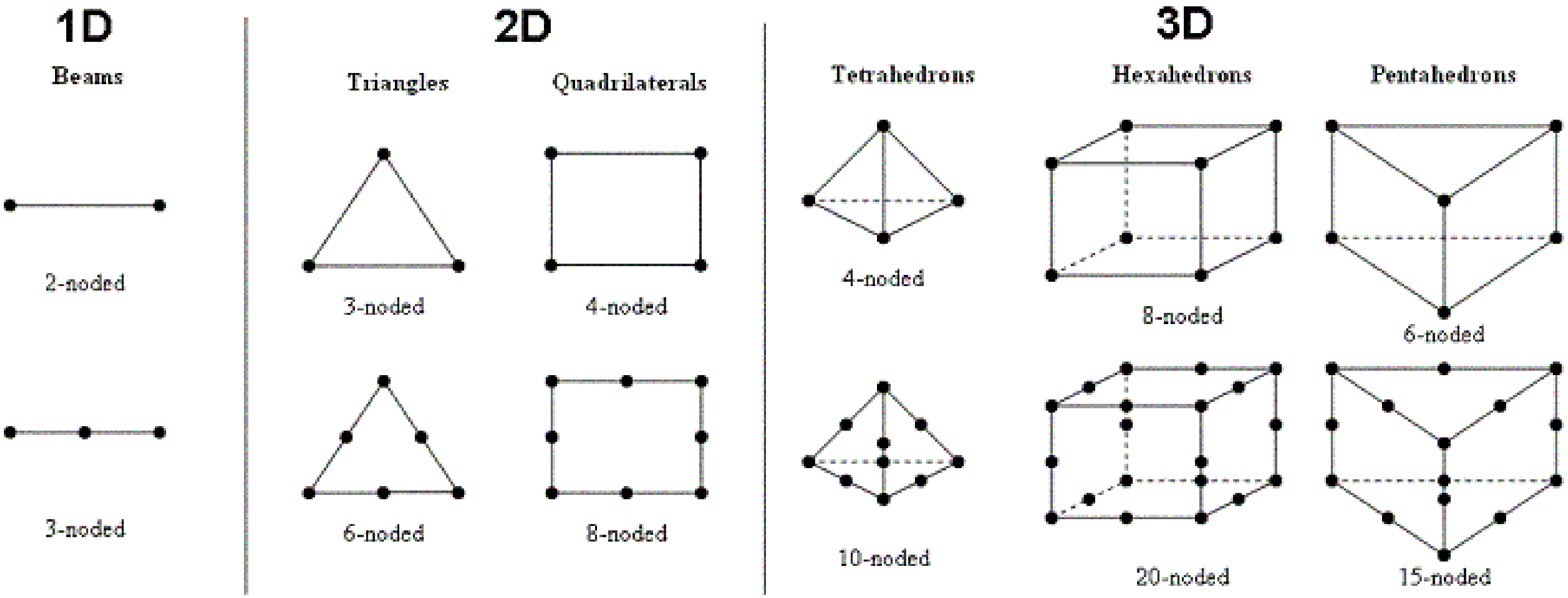
Consist in reducing the size of the elements in order to obtain a better approximation of the domain.



Solution convergence strategies

- **P-method**

Consist in increasing the order of the interpolation shape functions (use more nodes) in order to obtain a better approximation of the deformation



Solution convergence strategies

- **Which method is better?** There's no a unique conclusion
  - In the p-version of the finite element method the rate of convergence cannot be slower than in the h-version.
  - Numerical oscillation problem would happen for p-version of the finite element method.
  - For obvious practical reasons, finite element analyses should be **both efficient and reliable**.

- **As a general point of view:**
  - For structural analysis, h-method is more popular. Two-order element is a good application considering the efficiency and reliability.
  - p-method seems to act against the original goal of FEM