

Mètodes Numèrics:

A First Course on Finite Elements

Interpolation (Shape Functions)

Following: *Curs d'Elements Finitos amb Aplicacions* (J. Masdemont)

<http://hdl.handle.net/2099.3/36166>

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Interpolation Problem

The general solution of problem using Finite Elements has the following steps:

Finite Elements

Stp1: Discretize in **elements**

Stp2: Write the **variational** equations

Stp3: Build the Linear System
Impose the BC

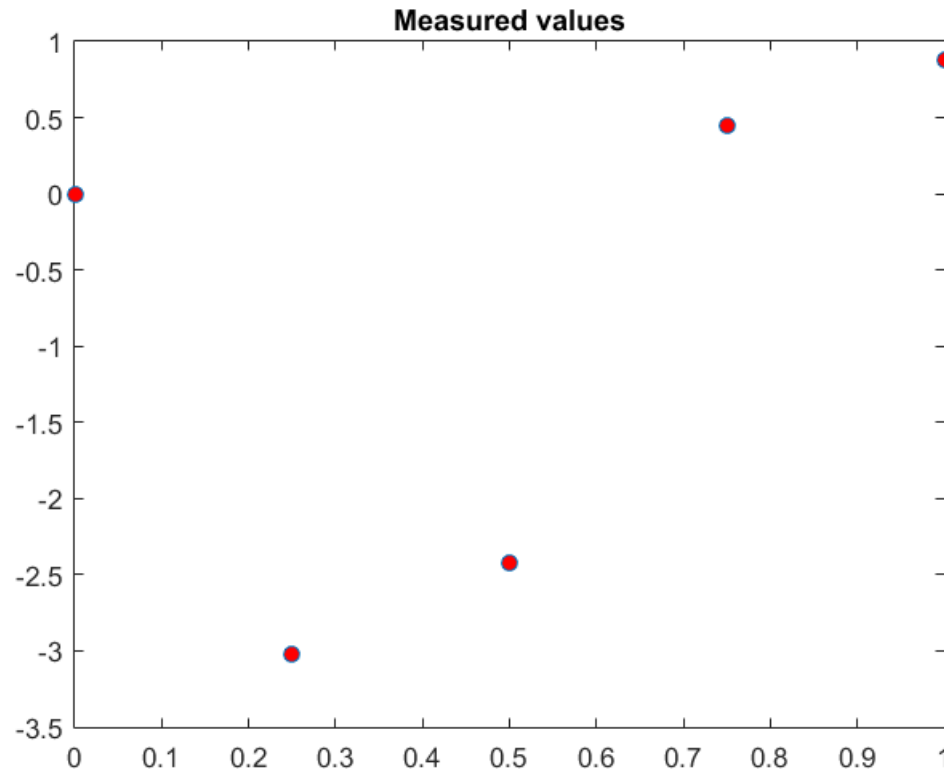
Stp4: Get **nodes** solution

Stp5: Extent solution to the Domain



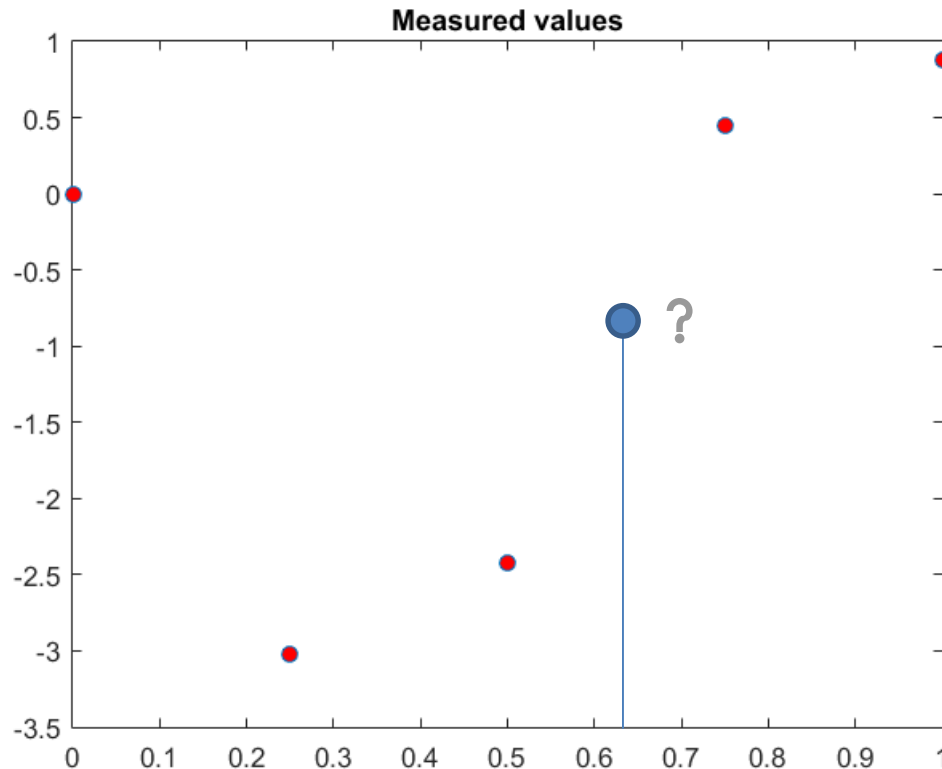
Interpolation Problem 1D

Let's assume that we have **measured** an unknown magnitude $u = u(x)$ on a set of data points $x = [x_1, x_2, \dots, x_N]$ obtaining a set of values $u = [u_1, u_2, \dots, u_N]$



Interpolation Problem 1D

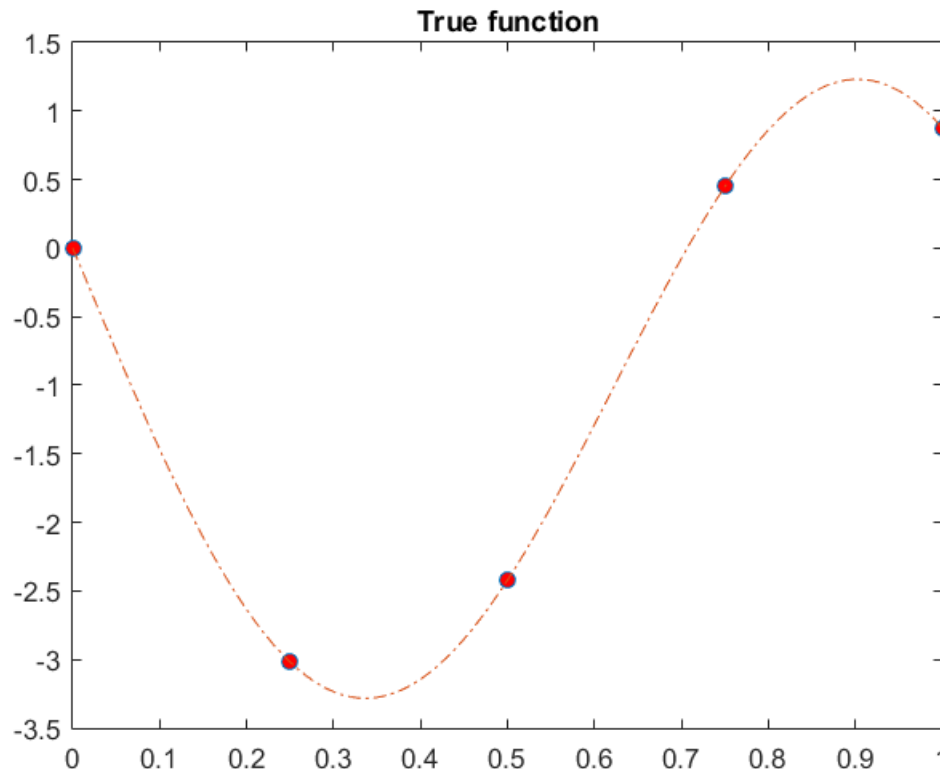
Let's assume that we have **measured** an unknown magnitude $u = u(x)$ on a set of data points $x = [x_1, x_2, \dots, x_N]$ obtaining a set of values $u = [u_1, u_2, \dots, u_N]$



How can we guess
the value of a new point?

Interpolation Problem 1D

Let's assume that we have **measured** an unknown magnitude $u = u(x)$ on a set of data points $x = [x_1, x_2, \dots, x_N]$ obtaining a set of values $u = [u_1, u_2, \dots, u_N]$



Actual function

$$f(x) = x^3 - 3x^2 - 3\sin(5x)$$

Interpolation Problem 1D

To **Interpolate** a set of data measured points (x_i, u_i) means to build a function $P_n(x)$ (usually a **polynomial**) passing through these points. That is

$$u_i = P_n(x_i), \quad i = 1 \dots N$$

Theorem:

The interpolation problem **has a unique** solution when $n = N - 1$.

Idea of the proof:

$$P_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \longrightarrow n + 1 \equiv N \text{ unknowns}$$

$$u_i = P_n(x_i), \quad i = 1 \dots N \longrightarrow N \text{ linear equations}$$

Unique solution if the system is compatible determined. That means the interpolation polynomial has **degree one less** than the number of points \square



Shape Functions 1D

Shape Functions 1D

Lagrange Polynomials (Shape functions)

One (N-1) degree Polynomial for each measure point x_i , $i = 1, \dots, N$

$$\psi_i(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_{N-1})(x - x_N)}{(x_i - x_1)(x_i - x_2) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_{N-1})(x_i - x_N)}$$

Properties:

1.- Lagrange Polynomials are **a base** of the set $\mathbb{P}_n[x]$

2.- $\psi_i(x_j) = 0$ if $i \neq j$

3.- $\psi_i(x_i) = 1$

Interpolation Polynomial (a linear combination of the shape functions)

$$P_n(x) = u_1\psi_1(x) + u_2\psi_2(x) + \dots + u_N\psi_N(x)$$

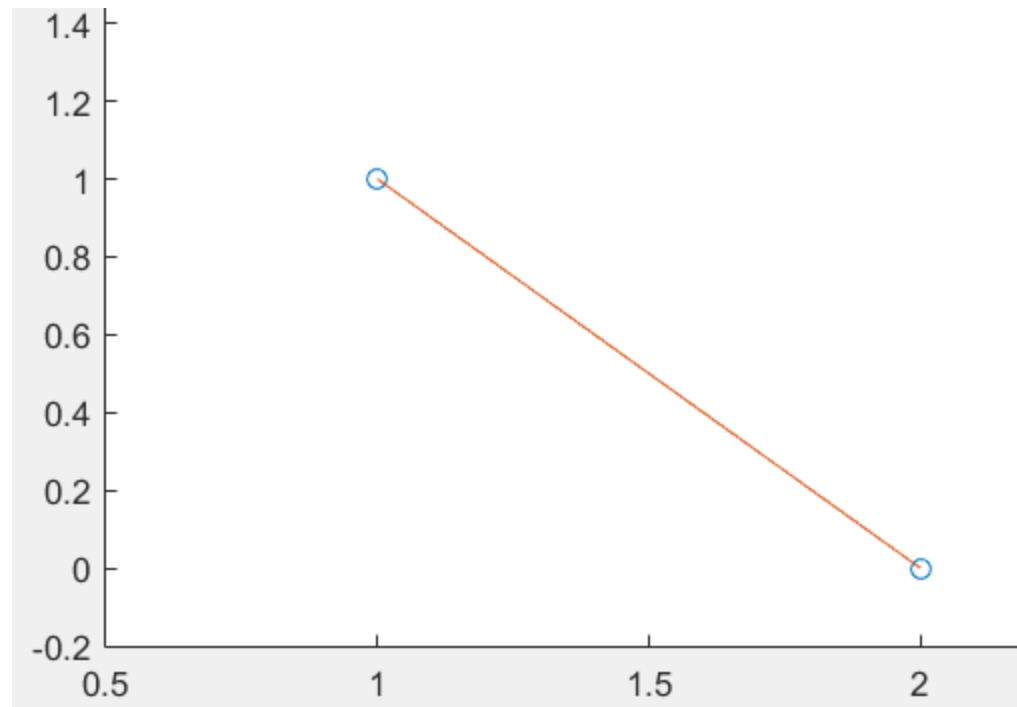
Shape Functions 1D

N = 2 only two points: $x = [x_1, x_2]$ and their measures $u = [u_1, u_2]$

$$\psi_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}$$

$$x = [1, 2]$$

It is a **straight line**



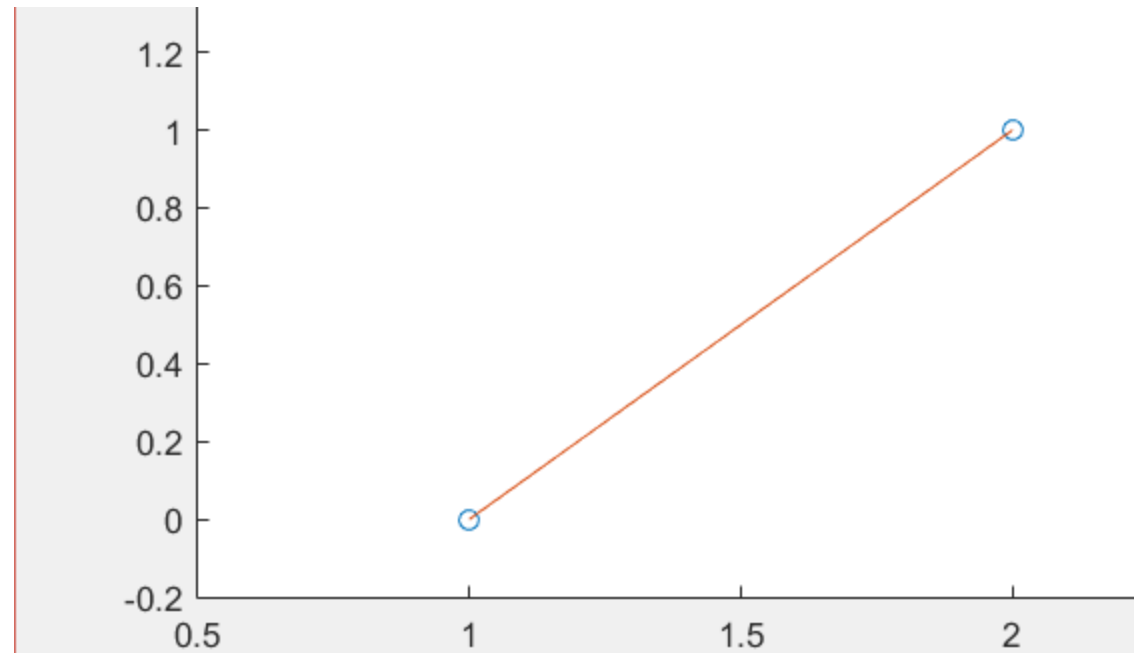
Shape Functions 1D

N = 2 only two points: $x = [x_1, x_2]$ and their measures $u = [u_1, u_2]$

$$\psi_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}$$

$$\psi_2(x) = \frac{(x - x_1)}{(x_2 - x_1)}$$

$x = [1, 2]$



Shape Functions 1D

N = 2 only two points: $x = [x_1, x_2]$ and their measures $u = [u_1, u_2]$

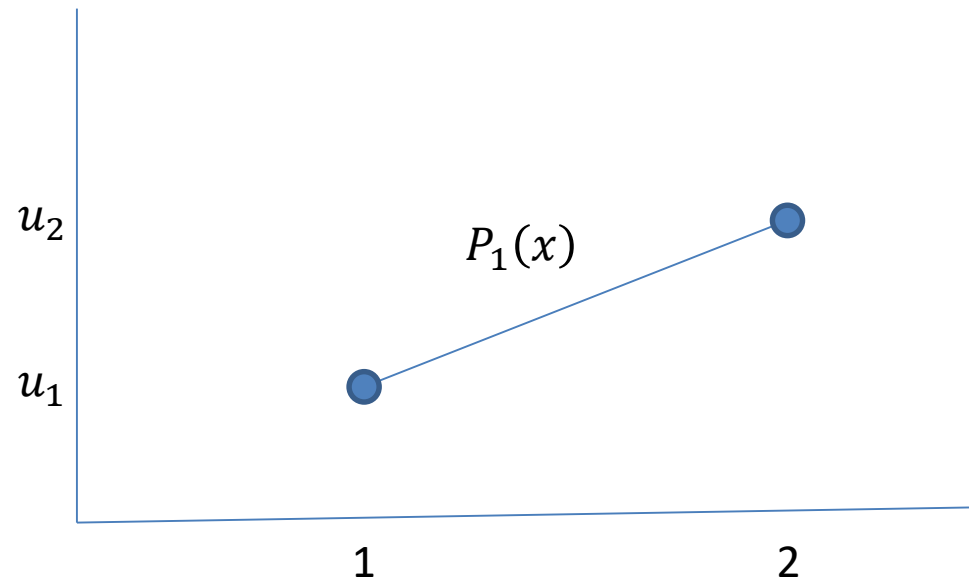
$$\psi_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}$$

$$\psi_2(x) = \frac{(x - x_1)}{(x_2 - x_1)}$$

Interpolation Polynomial

$$P_1(x) = u_1\psi_1(x) + u_2\psi_2(x)$$

It is **also a straight line** (linear interpolation)



Shape Functions 1D

N = 2 only two points: $x = [x_1, x_2]$ and their measures $u = [u_1, u_2]$

$$\psi_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}$$

$$\psi_2(x) = \frac{(x - x_1)}{(x_2 - x_1)}$$

Interpolation Polynomial

$$P_1(x) = u_1\psi_1(x) + u_2\psi_2(x)$$

Example:

In the case $x = [1, 2]$ and $u = [0.7, 1.8]$

$$\psi_1(x) = \frac{(x - 2)}{(1 - 2)} = -(x - 2)$$

$$\psi_2(x) = \frac{(x - 1)}{(2 - 1)} = (x - 1)$$

How to approximate the value for $x = 1.5$?

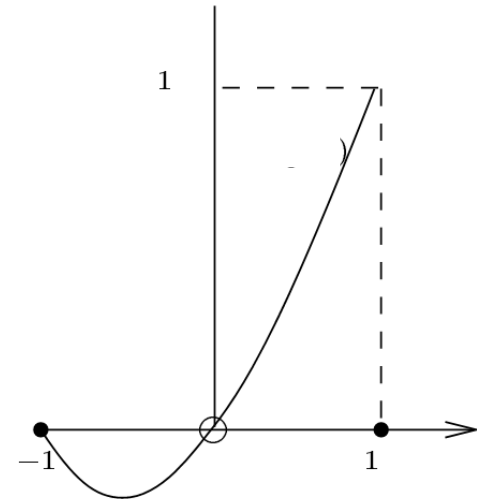
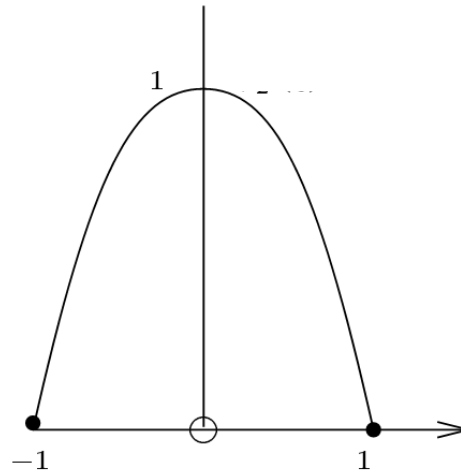
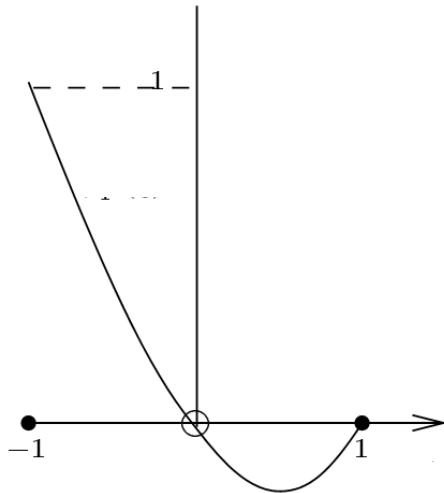
$$\begin{aligned} P_1(1.5) &= 0.7\psi_1(1.5) + 1.8\psi_2(1.5) \\ &= 0.7 * 0.5 + 1.8 * 0.5 = \mathbf{1.25} \end{aligned}$$

Shape Functions 1D

N = 3 three points: $x = [x_1, x_2, x_3]$ and measures $u = [u_1, u_2, u_3]$

$$\psi_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \quad \psi_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$\psi_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$



Shape Functions 1D

N = 3 three points: $x = [x_1, x_2, x_3]$ and measures $u = [u_1, u_2, u_3]$

$$\psi_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \quad \psi_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$\psi_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Interpolation Polynomial

$$P_2(x) = u_1\psi_1(x) + u_2\psi_2(x) + u_3\psi_3(x)$$

Now it is a **parabola (quadratic interpolation)**

Shape Functions 1D

Example:

Consider $x = [-1, 0, 1]$ and $u = [1.2, 0.7, 1.8]$. Compute $u(0.2)$?

$$\psi_1(x) = \frac{(x - 0)(x - 1)}{(-1 - 0)(-1 - 1)} = \frac{1}{2}x(x - 1)$$

$$\psi_2(x) = \frac{(x + 1)(x - 1)}{(0 + 1)(0 - 1)} = -(x + 1)(x - 1)$$

$$\psi_3(x) = \frac{(x + 1)(x - 0)}{(1 + 1)(1 - 0)} = \frac{1}{2}x(x + 1)$$

$$P_2(0.2) = 1.2\psi_1(0.2) + 0.7\psi_2(0.2) + 1.8\psi_3(0.2) = 0.7920$$

Shape Functions 1D

- Matlab code:

```
x = 0:0.25:1; % measure points
f = @(x) x.^3-3*x.^2-3*sin(5*x); %inline exemple function
y = f(x);      %measured values
figure()
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
title('Measured values'); % plot only measured values
figure()
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
hold on;
xx = 0:0.01:1; % take many points for a better plot
yy = f(xx);
plot(xx, yy,'-.')
title('True function'); % plot the function
hold off
```

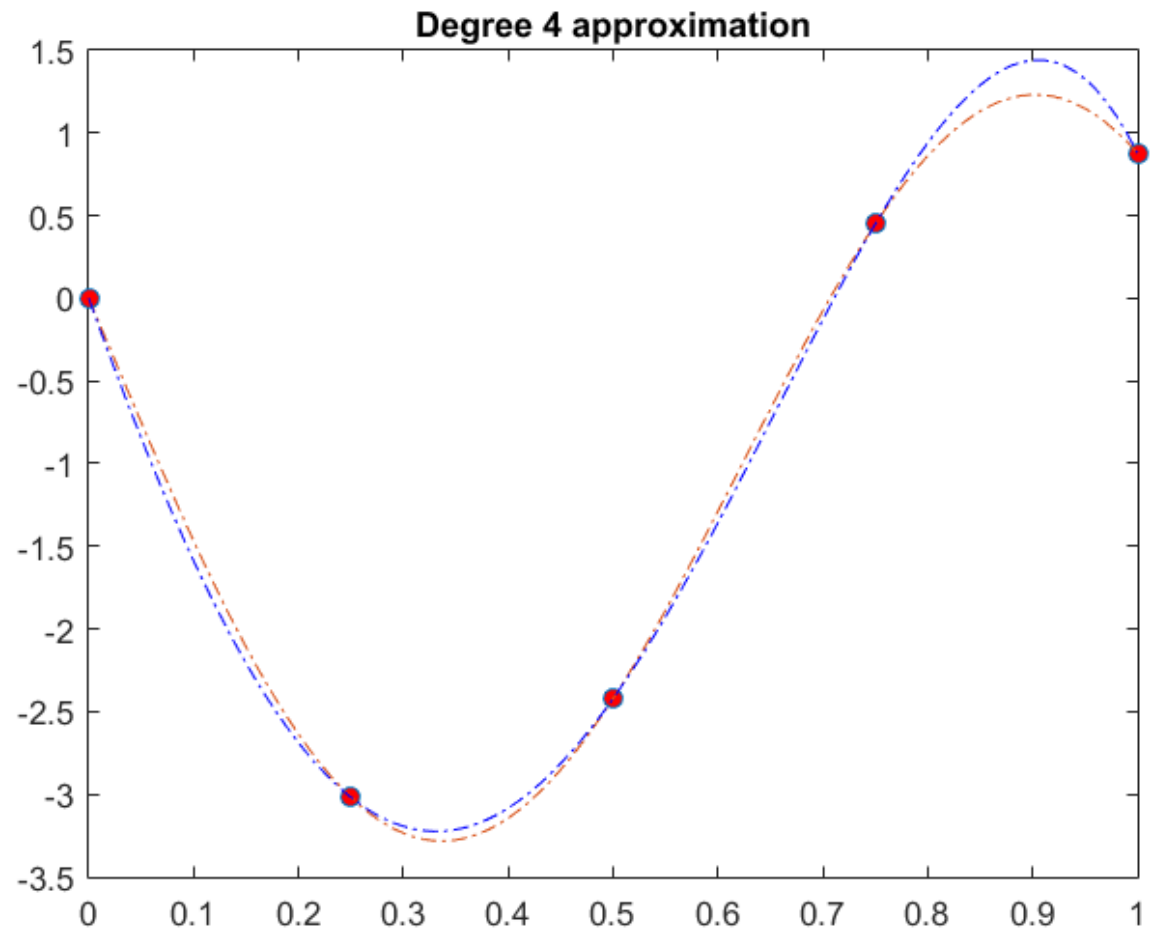
Shape Functions 1D

- Matlab code (continuation):

```
figure() % Polyfit degree 4 function (interpolator)
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
hold on;
plot(xx, yy, '-.')
p = polyfit(x,y,4);
yyy = polyval(p, xx);
plot(xx, yyy, '-.b')
title('Degree 4 approximation');
hold off
```

Shape Functions 1D

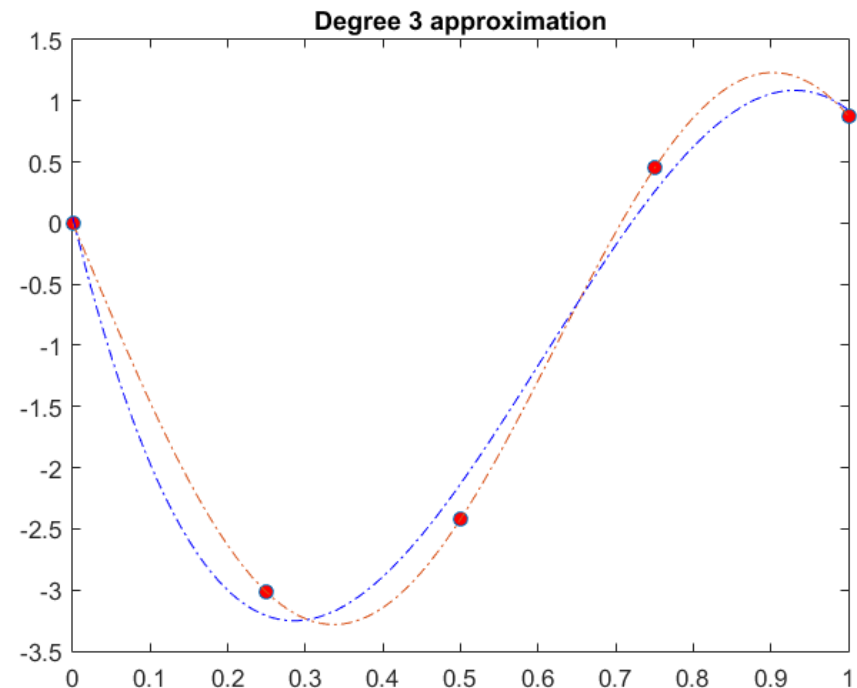
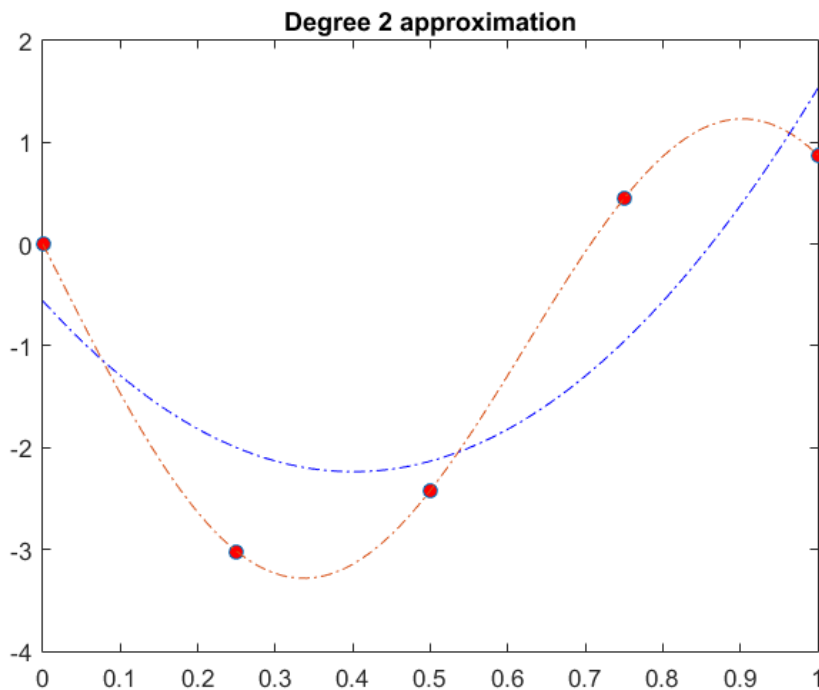
- **Interpolation Results:**



Shape Functions 1D

- **Approximation:** In case $n < (N - 1)$ there is NO solution (incompatible system) because the polynomial can not pass through all the points. In that case: **least square approximation**

$$P_n(x) = \min(\| \tilde{P}_n(x) - f(x) \|)$$

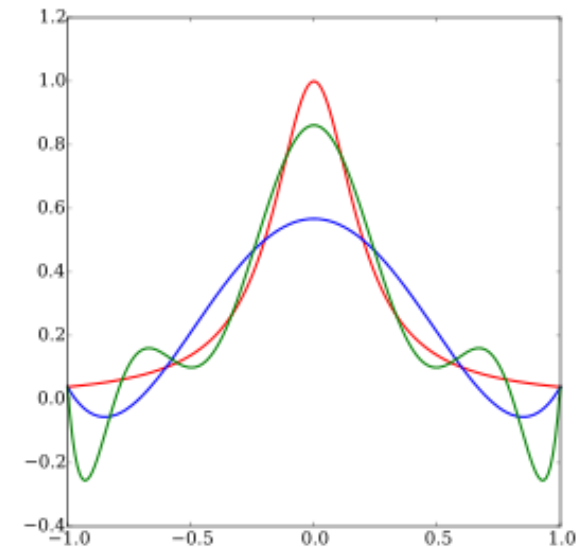


Shape Functions 1D

- **Higher interpolation:** When N is big (>8) there is a better solution than using a polynomial of high degree.
- **Runge's Phenomenon:**

$$f(x) = \frac{1}{1 + 25x^2}$$

- The **red curve** is the Runge function $f(x)$.
- The **blue curve** is a 5th-order interpolating polynomial (using six equally spaced interpolating points).
- The **green curve** is a 9th-order interpolating polynomial (using ten equally spaced interpolating points).



Shape Functions 1D

Two possible approaches:

Polygonal curve: Take the points in pairs and build a line segment (1st order interp) in each case. **Continuous but not derivable.**

Matlab code:

```
%% Polygonal
```

```
figure()
```

```
x=0:0.1:1;
```

```
y=f(x);
```

```
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
```

```
hold on;
```

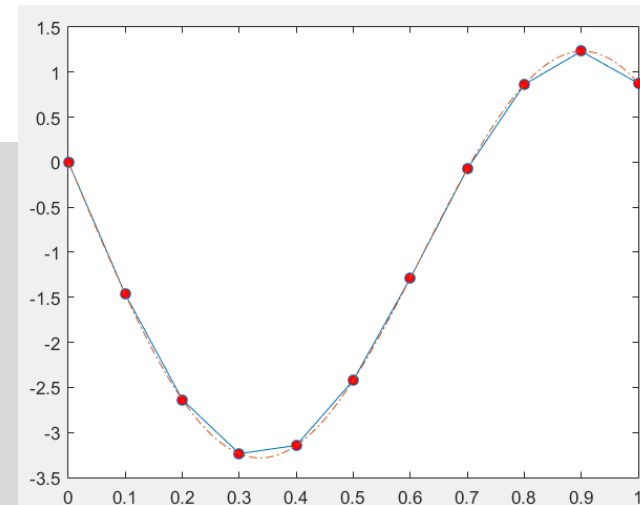
```
plot(xx,yy,'-.'
```

```
xxx=0:0.01:1;
```

```
yyy=interp1(x,y,xxx); %only substitution is possible
```

```
plot(xxx,yyy,'b')
```

```
hold off
```



Shape Functions 1D

Two possible approaches:

Spline curve (cubic): Take the points in groups of 2 and build a third order Polynomial (cubic interp) in each case imposing C^2 **continuity**.

Matlab code (continuation):

```
%% Spline
```

```
figure()
```

```
x=0:0.1:1;
```

```
y=f(x);
```

```
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
```

```
hold on;
```

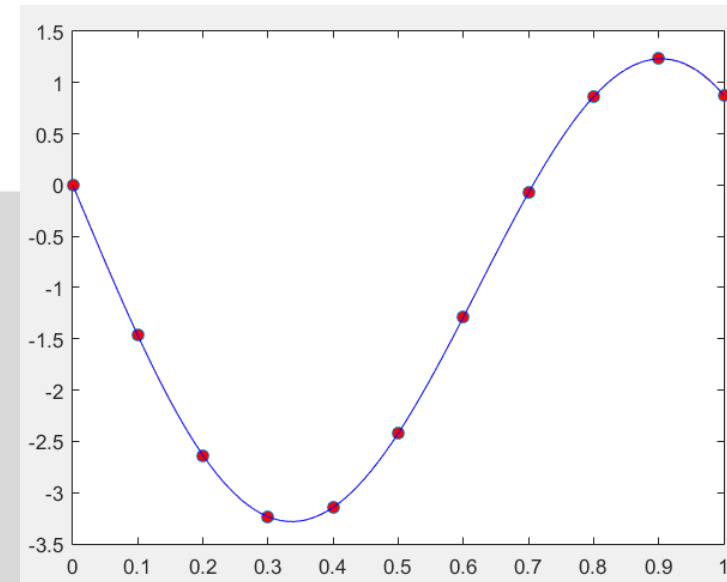
```
plot(xx,yy,'-.'
```

```
xxx = 0:0.01:1;
```

```
yyy = spline(x,y,xxx); %only substitution is possible
```

```
plot(xxx,yyy,'b')
```

```
hold off
```





Shape Functions 2D

Shape Functions 2D

How can we extend the interpolation problem to 2D?

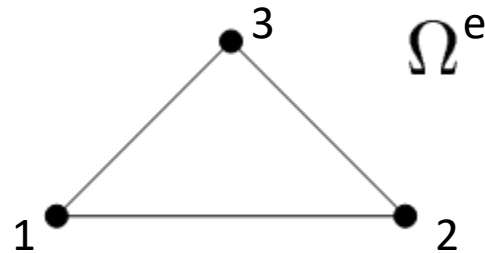
That is, how can we approximate the value of $f(x, y)$ using a 2D polynomial $P_n(x, y)$?

We restrict to fixed shapes:

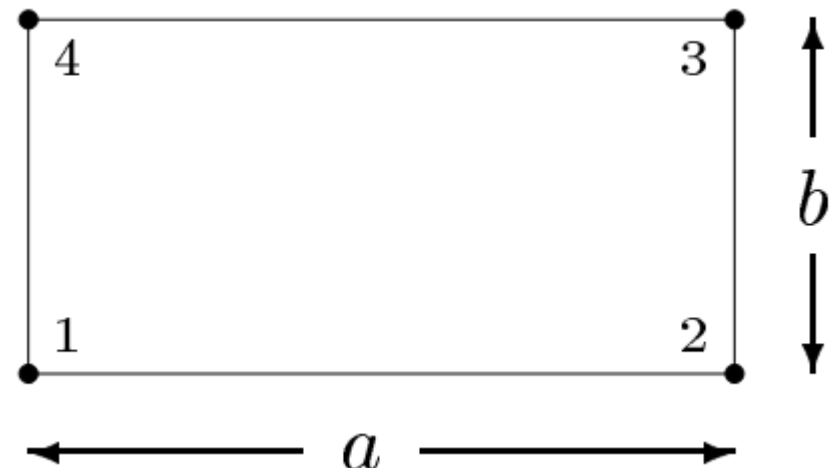
Triangular element

Numbered

counterclockwise



Quadrilateral element



Triangular Shape Functions

We need the equivalent to the Lagrange polynomials (2-dim)

$$\psi_i^e(x, y) = c_1 + c_2x + c_3y \quad (\text{three unknowns})$$

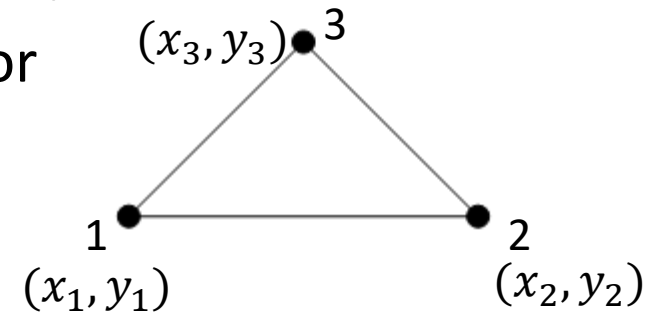
such that, it's value is 1 for vertex i , and 0 for the other two:

For the first vertex:

$$1 = \psi_1^e(x_1, y_1) = c_1 + c_2x_1 + c_3y_1$$

$$0 = \psi_1^e(x_2, y_2) = c_1 + c_2x_2 + c_3y_2$$

$$0 = \psi_1^e(x_3, y_3) = c_1 + c_2x_3 + c_3y_3$$



$$\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \longrightarrow \quad \mathbf{c} = \mathbf{A} \setminus \mathbf{b} \quad (\text{matlab solution})$$

Triangular Shape Functions

Explicitly

$$\psi_i^e(x, y) = c_{1,i} + c_{2,i}x + c_{3,i}y, \quad i = 1, 2, 3$$

$$c_{1,i} = \frac{x_j y_k - x_k y_j}{2A_e}$$

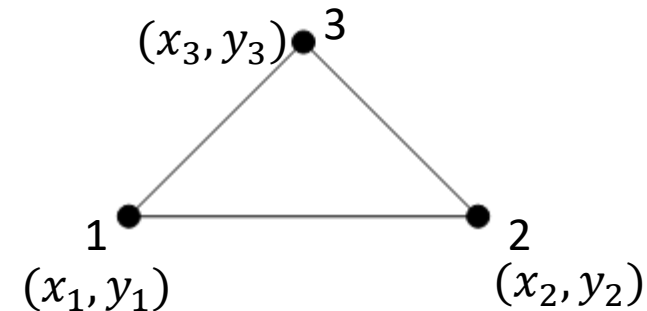
$$c_{2,i} = \frac{y_j - y_k}{2A_e}$$

$$c_{3,i} = \frac{x_k - x_j}{2A_e}$$

where $(i, j, k) = (1, 2, 3)$ cyclic and A_e is the **Area** of Ω_e

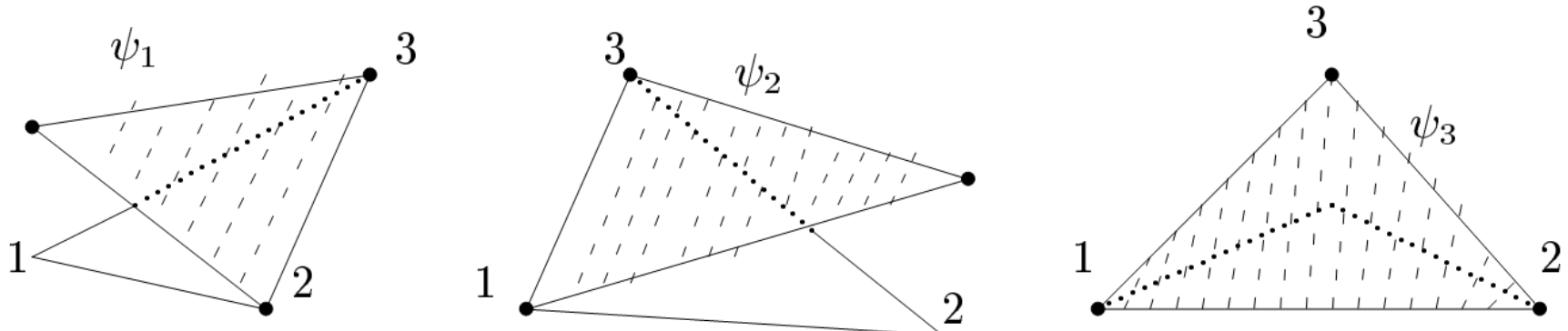
Remember:

$$\text{Area} = \frac{1}{2} \det \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$



Triangular Shape Functions

Once we have the shape functions $\psi_i^e(x, y)$, $i = 1, 2, 3$
(geometrically they are planes passing through one of the edges)



finally, the interpolation value of a point (\tilde{x}, \tilde{y}) is

$$f(\tilde{x}, \tilde{y}) = f(x_1, y_1)\psi_1^e(\tilde{x}, \tilde{y}) + f(x_2, y_2)\psi_2^e(\tilde{x}, \tilde{y}) + f(x_3, y_3)\psi_3^e(\tilde{x}, \tilde{y})$$

The terms

$\alpha_i^e = \psi_i^e(\tilde{x}, \tilde{y})$ are known as **Barycentric Coordinates** of (\tilde{x}, \tilde{y})

(We'll see more in the practical sessions)

Triangular Shape Functions

Example: The temperature of a thin triangular plate has been measured at the vertices $T(v_1) = 40^\circ$, $T(v_2) = 90^\circ$, $T(v_3) = 10^\circ$.

If the coordinates of the vertices are: $v_1 = (0,0)$, $v_2 = (5,0)$, $v_3 = (2,3)$, estimate the temperature at the point $p = (3,1)$?

Solution:

$$\begin{aligned}\psi_1^e(x, y) &= 1 - 0.2x - 0.2y & \longrightarrow & \alpha_1 = \psi_1^e(3,1) \\ \psi_2^e(x, y) &= 0 + 0.2x - 0.1333y & \longrightarrow & \alpha_2 = \psi_2^e(3,1) \\ \psi_3^e(x, y) &= 0 - 0x + 0.3333y & \longrightarrow & \alpha_3 = \psi_3^e(3,1)\end{aligned}$$

$$T(p) = 40\alpha_1 + 90\alpha_2 + 10\alpha_3 = 53.3333^\circ$$

Exercise: Compute the shape functions associated to the triangle $v_1 = (1,1)$, $v_2 = (3,2)$, $v_3 = (2,4)$.

Solution:

$$\begin{aligned}\psi_1^e(x, y) &= \frac{1}{5}(8 - 2x - y) \\ \psi_2^e(x, y) &= \frac{1}{5}(-2 + 3x - y) \\ \psi_3^e(x, y) &= \frac{1}{5}(-1 - x + 2y)\end{aligned}$$

Quadrilateral Shape Functions

A similar situation is found when a **rectangular** element is used.

$$\psi_i^e(x, y) = c_1 + c_2x + c_3y + c_4xy \quad (\text{four unknowns})$$

such that, its value is 1 for vertex i , and 0 for the other three:

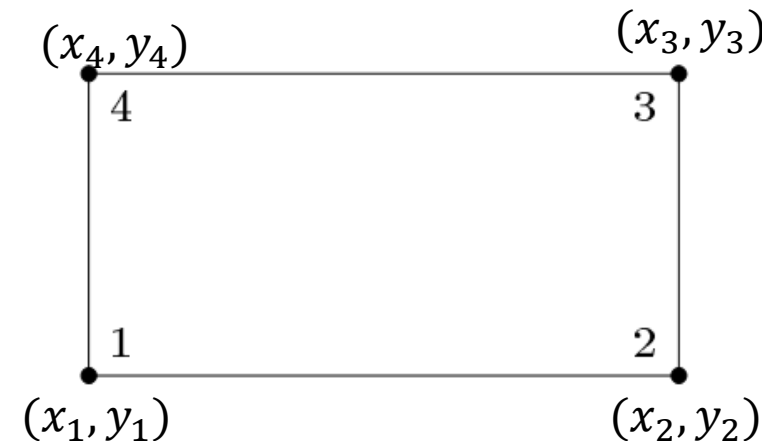
For the first vertex:

$$1 = \psi_1^e(x_1, y_1) = c_1 + c_2x_1 + c_3y_1 + c_4x_1y_1$$

$$0 = \psi_1^e(x_2, y_2) = c_1 + c_2x_2 + c_3y_2 + c_4x_2y_2$$

$$0 = \psi_1^e(x_3, y_3) = c_1 + c_2x_3 + c_3y_3 + c_4x_3y_3$$

$$0 = \psi_1^e(x_4, y_4) = c_1 + c_2x_4 + c_3y_4 + c_4x_4y_4$$



Analogously for the other vertices

(This is not correct for general quadrilateral elements)

Quadrilateral Shape Functions

General expressions for **Rectangles**:

If we denote by

$$a = x_2 - x_1$$

$$b = y_2 - y_1$$

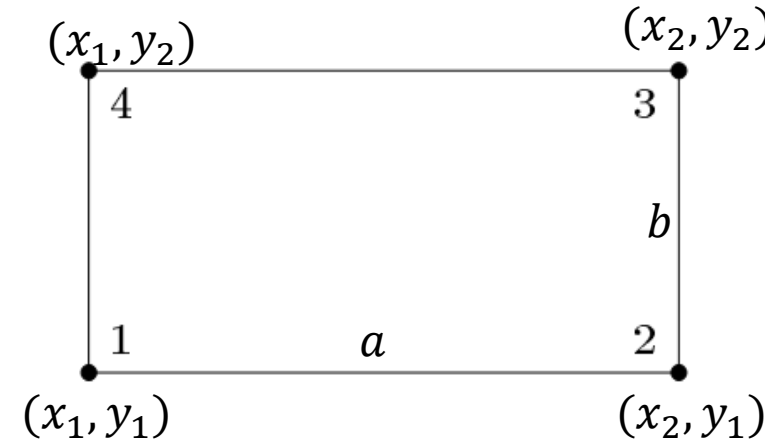
Then

$$\psi_1^e(x, y) = \frac{1}{ab} (x - x_2)(y - y_2)$$

$$\psi_2^e(x, y) = \frac{-1}{ab} (x - x_1)(y - y_2)$$

$$\psi_3^e(x, y) = \frac{1}{ab} (x - x_1)(y - y_1)$$

$$\psi_4^e(x, y) = \frac{-1}{ab} (x - x_2)(y - y_1)$$



Quadrilateral Shape Functions

The **reference** bilinear quadrilateral element Ω^R : $[-1,1] \times [-1,1]$

We can use matlab to compute the coeff.

$$\psi_j^e(x_i, y_i) = c_{1j} + c_{2j}x_i + c_{3j}y_i + c_{4j}x_iy_i = \delta_{ij}$$

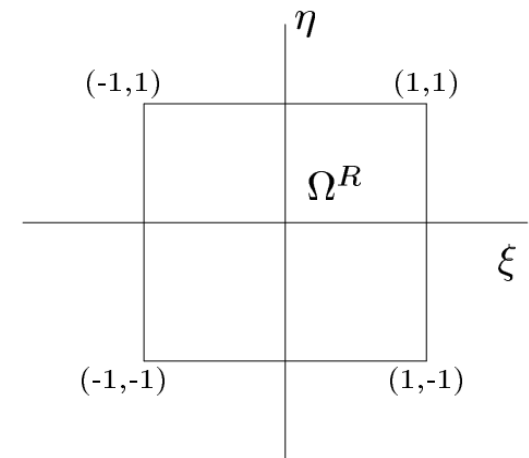
```
punts=[
-1,-1;
 1,-1;
 1,1;
-1,1];
```

```
A=[ones(4,1), punts(:,1), punts(:,2), punts(:,1).*punts(:,2)]
```

```
b=[1;0;0;0];
```

```
C1= A\b %coeff of the first shape function
```

```
% exercise: do it for the other shape functions
```



Quadrilateral Shape Functions

The shape functions for the reference element Ω^R , are:

$$\psi_1^R(\xi, \eta) = \frac{(1 - \xi)(1 - \eta)}{2} \cdot \frac{(1 - \eta)}{2},$$

$$\psi_2^R(\xi, \eta) = \frac{(1 + \xi)(1 - \eta)}{2} \cdot \frac{(1 - \eta)}{2},$$

$$\psi_3^R(\xi, \eta) = \frac{(1 + \xi)(1 + \eta)}{2} \cdot \frac{(1 + \eta)}{2},$$

$$\psi_4^R(\xi, \eta) = \frac{(1 - \xi)(1 + \eta)}{2} \cdot \frac{(1 + \eta)}{2}.$$

Or equivalently:

$$\psi_1^R(\xi, \eta) = \frac{1 - \xi - \eta + \xi\eta}{4}$$

$$\psi_2^R(\xi, \eta) = \frac{1 + \xi - \eta - \xi\eta}{4}$$

$$\psi_3^R(\xi, \eta) = \frac{1 + \xi + \eta + \xi\eta}{4}$$

$$\psi_4^R(\xi, \eta) = \frac{1 - \xi + \eta - \xi\eta}{4}$$

Quadrilateral Shape Functions

Example: The temperature of at the vertices of the **reference** quadrilateral has been measured as $T(v_1) = 10^\circ$, $T(v_2) = 20^\circ$, $T(v_3) = 30^\circ$, $T(v_4) = 40^\circ$.

Estimate the temperature at the point $p = (-0.3, 0.5)$?

Solution:

$$\psi_1^R(p) = 0.1625; \quad \psi_2^R(p) = 0.0875;$$

$$\psi_3^R(p) = 0.2625; \quad \psi_4^R(p) = 0.4875;$$

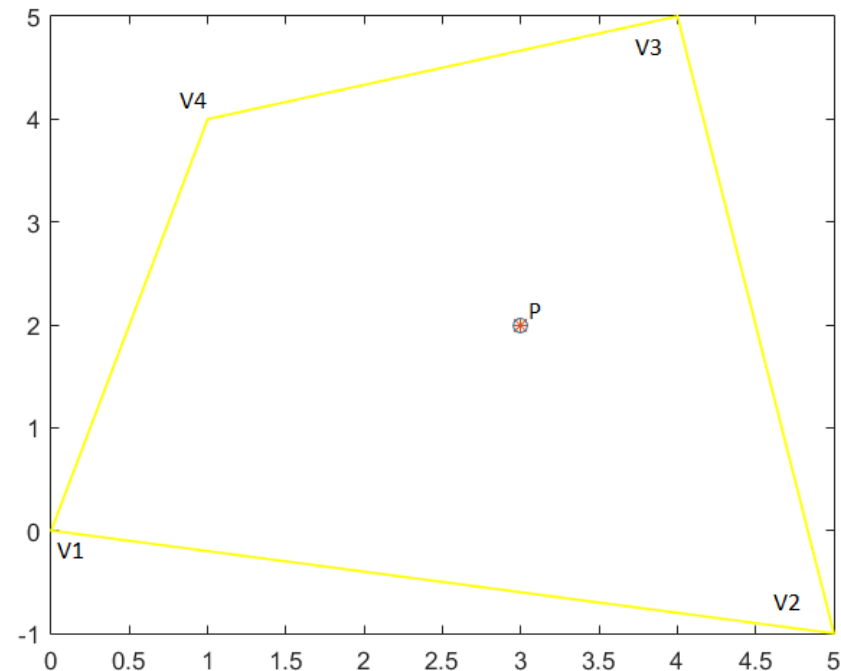
$$\begin{aligned} T(p) &= T_1\psi_1^R(p) + T_2\psi_2^R(p) + T_3\psi_3^R(p) + T_4\psi_4^R(p) \\ &= 30.75^\circ \end{aligned}$$

Quadrilateral Shape Functions

- For a General Quadrilateral element usually the shape function is NOT a 2 degree polynomial. Because of that it is not easy to compute these functions, but we still can compute the **Barycentric Coordinates**.

How to compute

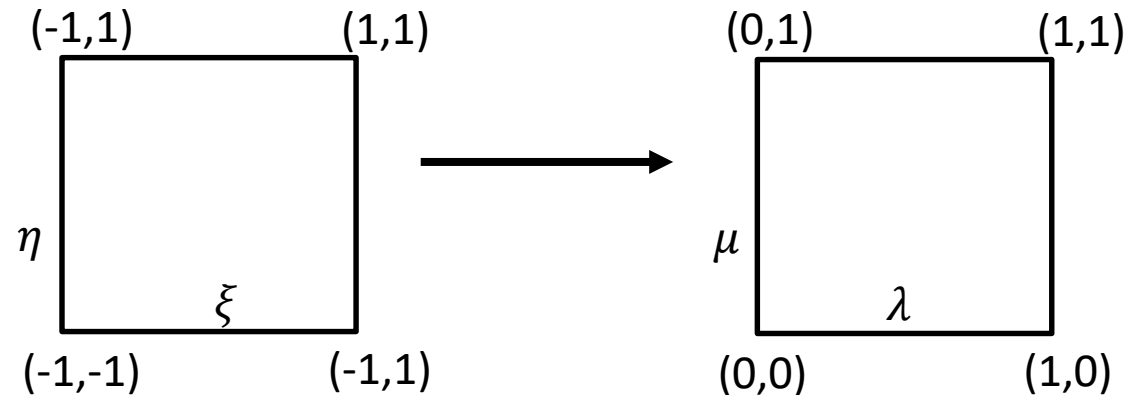
$$\alpha_i^k = \psi_i^k(P)?$$



Quadrilateral Shape Functions

- Change from the Reference quadrilateral to $[0,1] \times [0,1]$

$$\lambda = \frac{\xi + 1}{2}, \quad \mu = \frac{\eta + 1}{2}$$



Shape functions on $[0,1] \times [0,1]$:

$$\psi_1(\lambda, \mu) = (1 - \lambda)(1 - \mu)$$

$$\psi_3(\lambda, \mu) = \lambda\mu$$

$$\psi_2(\lambda, \mu) = \lambda(1 - \mu)$$

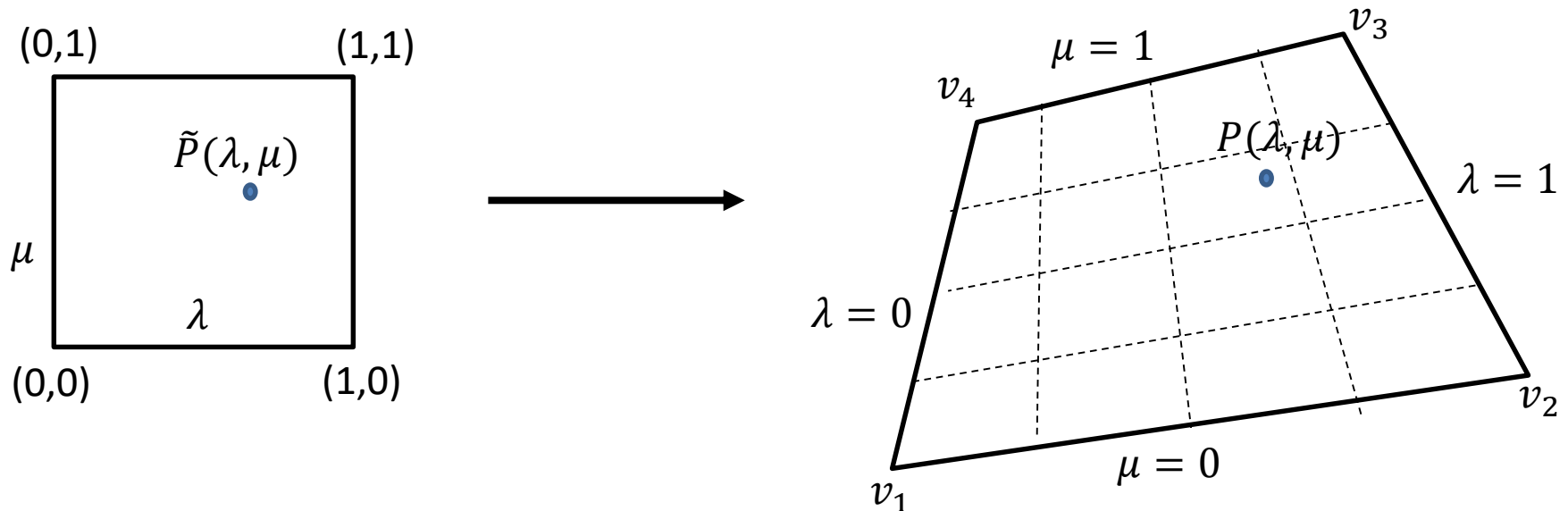
$$\psi_4(\lambda, \mu) = (1 - \lambda)\mu$$

Quadrilateral Shape Functions

- General Quadrilateral: **Isoparametric** transformation

$$(x, y) = \psi_1(\xi, \eta)v_1 + \psi_2(\xi, \eta)v_2 + \psi_3(\xi, \eta)v_3 + \psi_4(\xi, \eta)v_4$$

Change to another quadrilateral using the shape functions: for simplicity we will use here the rectangle $[0,1] \times [0,1]$:



$$P(\lambda, \mu) = \underbrace{(1 - \lambda)(1 - \mu)}_{\alpha_1} v_1 + \underbrace{\lambda(1 - \mu)}_{\alpha_2} v_2 + \underbrace{\lambda\mu}_{\alpha_3} v_3 + \underbrace{(1 - \lambda)\mu}_{\alpha_4} v_4$$

Quadrilateral Shape Functions

We can compute them using $\lambda, \mu \in [0,1]$, as a parametrization of the quadrilateral edges.

$$P = (1 - \lambda)(1 - \mu) v_1 + \lambda(1 - \mu) v_2 + \lambda\mu v_3 + (1 - \lambda)\mu v_4$$

Rearranging the terms, the previous equation can be written as:

$$\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c} + \lambda\mu\mathbf{d} = 0$$

where

$$\begin{aligned}\mathbf{a} &= v_1 - P, & \mathbf{b} &= v_2 - v_1, & \mathbf{c} &= v_4 - v_1, \\ \mathbf{d} &= v_1 - v_2 + v_3 - v_4\end{aligned}$$

Quadrilateral Shape Functions

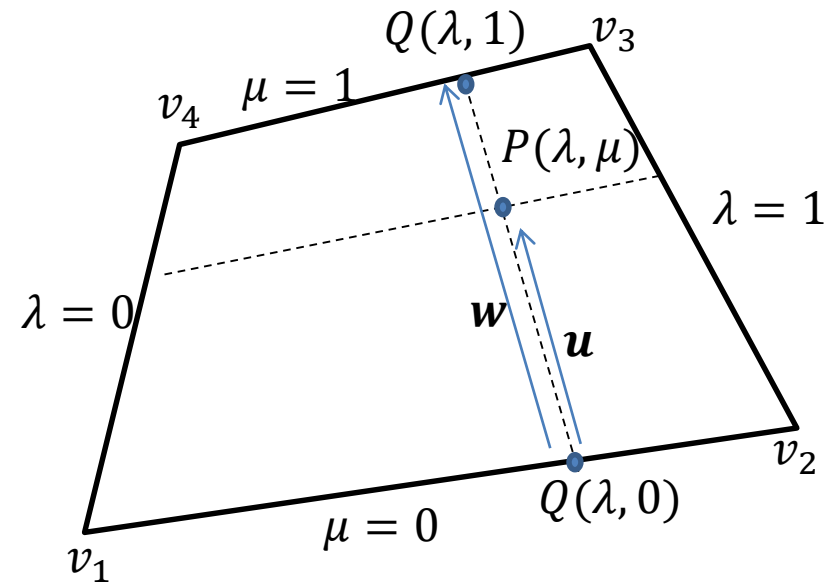
We impose that the vector

$$\mathbf{w} = \overrightarrow{Q(\lambda, 1) - Q(\lambda, 0)}$$

and the vector

$$\mathbf{u} = \overrightarrow{P - Q(\lambda, 0)}$$

are parallel, i.e. $\mathbf{w} \times \mathbf{u} = \mathbf{0}$



From the isometric transformation
we obtain:

$$\mathbf{w} = v_4 - v_1 + \lambda(v_3 - v_4 + v_1 - v_2) = \mathbf{c} + \lambda\mathbf{d}$$

$$\mathbf{u} = P - v_1 + \lambda(v_1 - v_2) = -(\mathbf{a} + \lambda\mathbf{b})$$

Quadrilateral Shape Functions

If this two vectors are parallel

$$(\mathbf{a} + \lambda \mathbf{b}) \times (\mathbf{c} + \lambda \mathbf{d}) = 0$$

or

$$(\mathbf{b} \times \mathbf{d})\lambda^2 + (\mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c})\lambda + \mathbf{a} \times \mathbf{c} = 0$$

which is a quadratic equation that can be solved for λ .

Hint: All the 2D vectors are extended with a zero third component in order to define the cross product.

Analogously a similar equation is found for μ .

Quadrilateral Shape Functions

- Example: Given a quadrilateral defined by vertices $v_1 = (0,0)$, $v_2 = (5, -1)$, $v_3 = (4,5)$, $v_4 = (1,4)$ compute the barycentric coordinates of point $p=(3,2)$.

Sol. $\lambda = 0.6250$, $\mu = 0.5$

$$\alpha = [0.1875 \quad 0.3125 \quad 0.3125 \quad 0.1875].$$

(examples using Matlab will be shown in practices)