UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria Industrial de Barcelona
ETSEIB

Mètodes Numèrics:

A First Course on Finite Elements

Numerical Integration

Dept. Matemàtiques ETSEIB - UPC BarcelonaTech

1D Gaussian quadrature

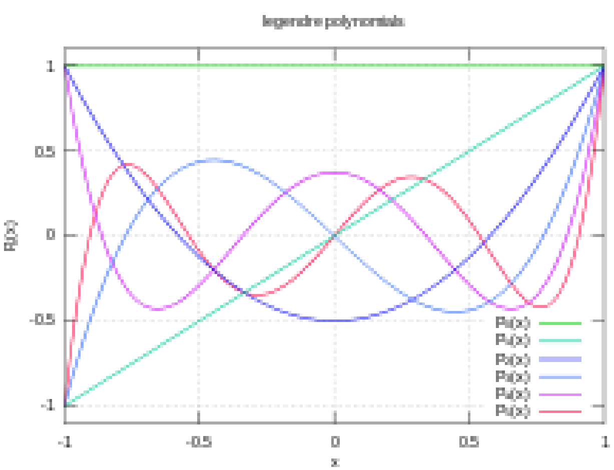
- **Gauss Methods**: Use strategic points (zeros of one special orthogonal *Legendre's polynomial*) in the [-1,1] domain (in our case it's appropriate because it is equivalent to the reference 1D element).

$$\int_a^b f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i)$$

- The formula is **exact** for polynomials of degree **2n-1**
- For general 1D domain [a,b] we use integral properties (change of variables) to pass computation to [-1,1]

1D Gaussian quadrature

If we fix the number of points. One can obtain their value (the zeros of the **Legendre Polynomials**) and the respective weights.



Number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{\frac{1}{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$
5	0	$\frac{128}{225}$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	$\frac{322-13\sqrt{70}}{900}$

https://en.wikipedia.org/wiki/Gaussian_quadrature

1D Gaussian quadrature

- Gaussian quadrature of order 1 (**one** point):

$$\int_{-1}^{1} g(\xi)\ \mathrm{d}\xi \approx 2g(0)$$

- Gaussian quadrature of order 2 (**two** points):

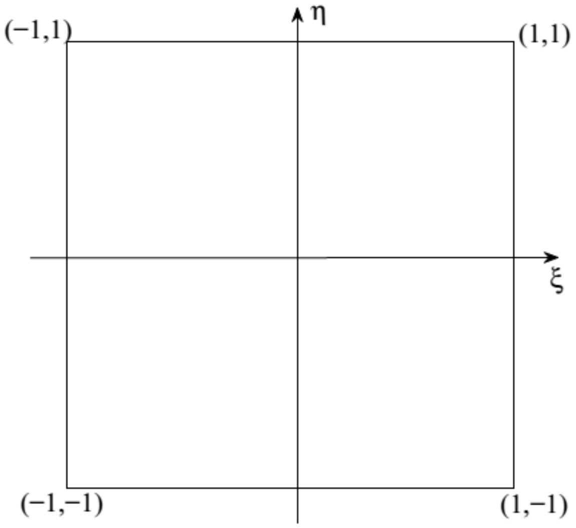
$$\int_{-1}^{1} g(\xi)\ \mathrm{d}\xi \approx g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

- Gaussian quadrature of order 3 (**three** points):

$$\int_{-1}^{1} g(\xi)\ \mathrm{d}\xi \approx \frac{5}{9} \cdot g\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + \frac{8}{9} \cdot g\left(0\right) + \frac{5}{9} \cdot g\left(\frac{\sqrt{3}}{\sqrt{5}}\right)$$

2D Gaussian quadrature

- Gaussian quadrature for the **reference quadrilateral element**



$$I = \iint_{R_{\rm st}} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta = \int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta.$$

- For a fixed η we can integrate with respect to ξ using 1D Gaussian quadrature

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta \approx \int_{-1}^{1}\left(\sum_{i=1}^{M} w_i g(\xi_i,\eta)\right)\mathrm{d}\eta,$$

where ξ_i and w_i are Gaussian quadrature points and weights of order M in the ξ direction

2D Gaussian quadrature

- Next integrating numerically with respect to η

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta \approx \sum_{i=1}^{M}\sum_{j=1}^{N} w_i \hat{w}_j g(\xi_i,\eta_j),$$

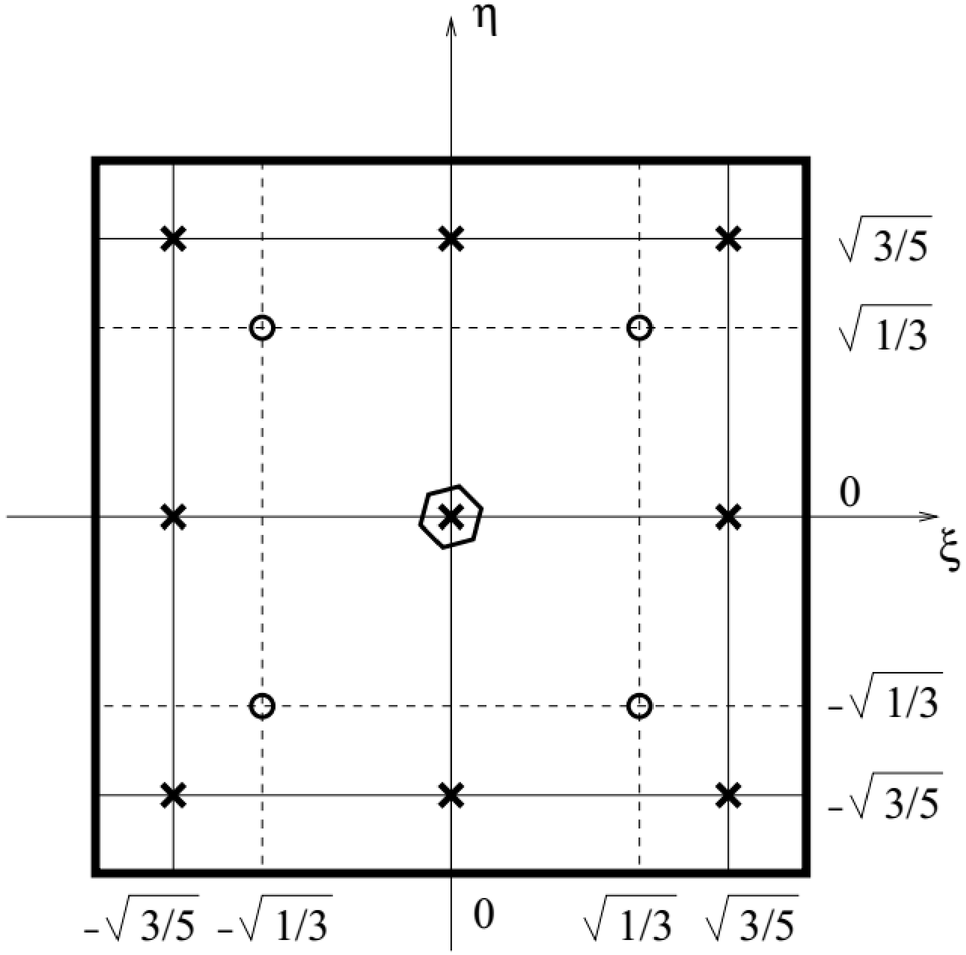
- Example N=M=3: (……….exact up to 5th degree in each variable)

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta \approx & \frac{5}{9}\cdot\frac{5}{9}\cdot g\left(-\frac{\sqrt{3}}{\sqrt{5}},-\frac{\sqrt{3}}{\sqrt{5}}\right)+\frac{8}{9}\cdot\frac{5}{9}\cdot g\left(0,-\frac{\sqrt{3}}{\sqrt{5}}\right)+\frac{5}{9}\cdot\frac{5}{9}\cdot g\left(\frac{\sqrt{3}}{\sqrt{5}},-\frac{\sqrt{3}}{\sqrt{5}}\right)\\ &+\frac{5}{9}\cdot\frac{8}{9}\cdot g\left(-\frac{\sqrt{3}}{\sqrt{5}},0\right)+\frac{8}{9}\cdot\frac{8}{9}\cdot g\left(0,0\right)+\frac{5}{9}\cdot\frac{8}{9}\cdot g\left(\frac{\sqrt{3}}{\sqrt{5}},0\right)\\ &+\frac{5}{9}\cdot\frac{5}{9}\cdot g\left(-\frac{\sqrt{3}}{\sqrt{5}},\frac{\sqrt{3}}{\sqrt{5}}\right)+\frac{8}{9}\cdot\frac{5}{9}\cdot g\left(0,\frac{\sqrt{3}}{\sqrt{5}}\right)+\frac{5}{9}\cdot\frac{5}{9}\cdot g\left(\frac{\sqrt{3}}{\sqrt{5}},\frac{\sqrt{3}}{\sqrt{5}}\right).\end{aligned}$$

2D Gaussian quadrature

- The idea is to take the points (ξ_i, η_j)

For the cases M=1,2,3 we obtain 1 point (hexagon), 4 points (circles) or 9 points (crosses) respectively.



2D Gaussian quadrature

```
% Gauss 2D from the 1D values
n=3; %exemple
[wx,ptx]=gaussValues(n);
[wy,pty]=gaussValues(n);
%points and weights
pt2D=[];
w=[];
for i=1:n
    for j=1:n
        pt2D=[pt2D;  [ptx(i),pty(j)]];
        w=[w ,  wx(i)*wy(j)];
    end
end
```

Gaussian pt2D (n=3)

-0,774596669241483	-0,774596669241483
-0,774596669241483	0
-0,774596669241483	0,774596669241483
0	-0,774596669241483
0	0
0	0,774596669241483
0,774596669241483	-0,774596669241483
0,774596669241483	0
0,774596669241483	0,774596669241483

Weights 2D (n=3)

0,308641975308642
0,493827160493827
0,308641975308642
0,493827160493827
0,790123456790123
0,493827160493827
0,308641975308642
0,493827160493827
0,308641975308642

2D Gaussian quadrature

- **Example:** Let's choose $g(\xi,\eta) = \xi^2\eta^2$

Analytical solution:

$$\int_{-1}^{1}\int_{-1}^{1}\xi^2\eta^2\ d\xi d\eta = \int_{-1}^{1}\xi^2\cdot\left[\frac{\eta^3}{3}\right]_{-1}^{1}\ d\xi = \frac{2}{3}\int_{-1}^{1}\xi^2 d\xi = \frac{4}{9}$$

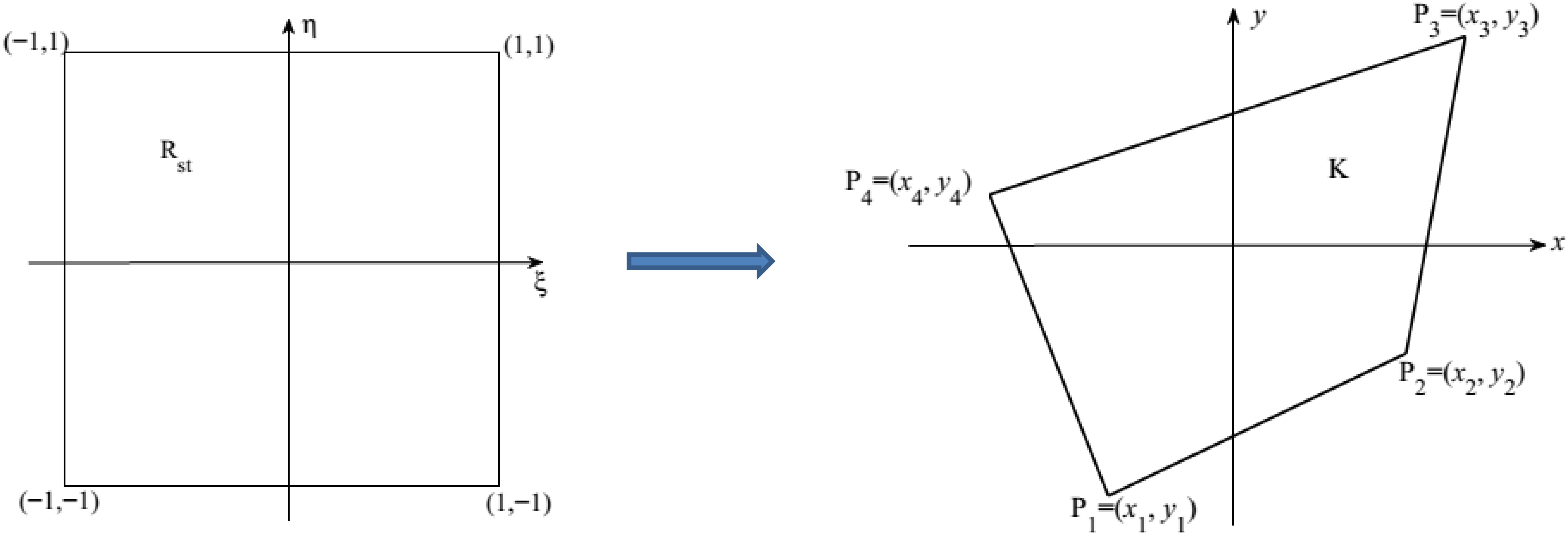
Numerical Solution:

```
n=2;
[w,pt]=gaussValues2D(n)
G=@(x,y) (x.^2).*y.^2;
Gxy=G(pt(:,1),pt(:,2));
Integ=sum(w'.*Gxy)
```

Sol: $0.4444 \cong \frac{4}{9}$

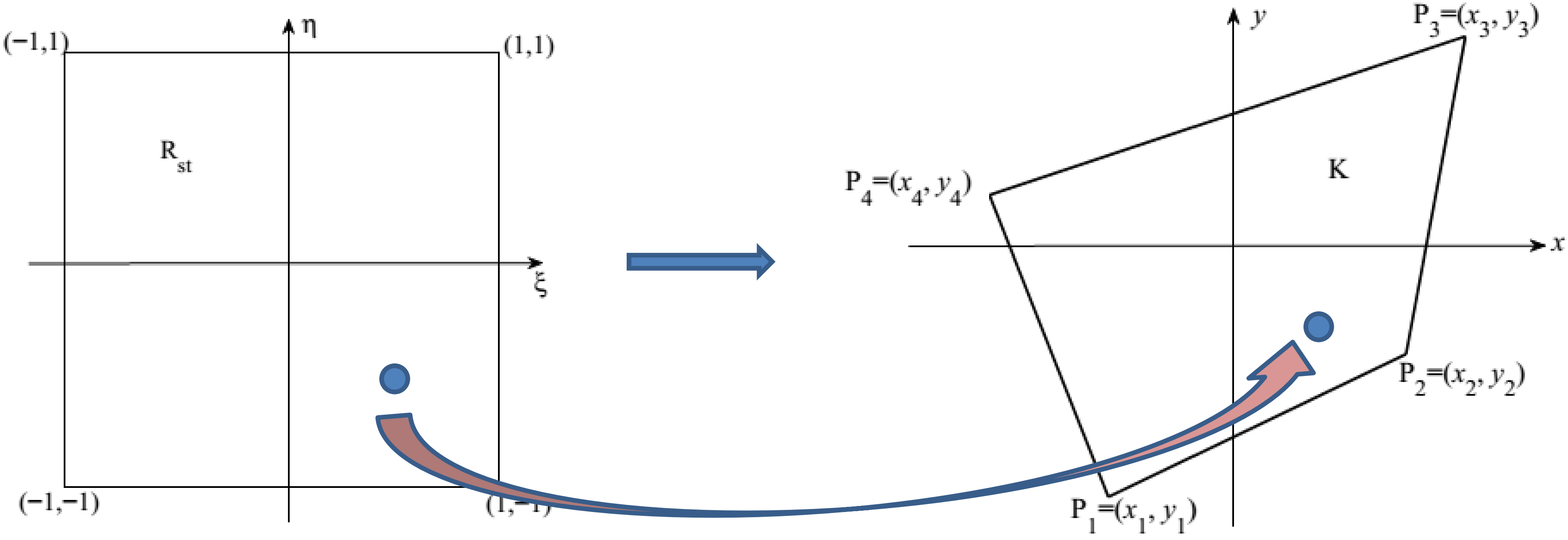
Quadrilateral Elements

- **General Quadrilateral Elements**



Quadrilateral Elements

- **General Quadrilateral Elements**



Change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \phi_k(\xi,\eta) = \begin{pmatrix} \psi_1^R(\xi,\eta)\,x_1^k + \psi_2^R(\xi,\eta)\,x_2^k + \psi_3^R(\xi,\eta)\,x_3^k + \psi_4^R(\xi,\eta)\,x_4^k \\ \psi_1^R(\xi,\eta)\,y_1^k + \psi_2^R(\xi,\eta)\,y_2^k + \psi_3^R(\xi,\eta)\,y_3^k + \psi_4^R(\xi,\eta)\,y_4^k \end{pmatrix}$$

Quadrilateral Elements

- **Integral over a General Quadrilateral Element**

Using the notation $\begin{array}{l} x = P(\xi,\eta) \\ y = Q(\xi,\eta) \end{array}$ for the change of variables

Then

$$\iint_K F(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{R_{\mathrm{st}}} F\left(P(\xi,\eta),Q(\xi,\eta)\right)|J(\xi,\eta)|\,\mathrm{d}\xi\mathrm{d}\eta,$$

where $J(\xi,\eta)$ is the Jacobian of the transformation

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{pmatrix}$$

and $|J(\xi,\eta)|$ is its **determinant**.

Quadrilateral Elements

- **The Jacobian term**

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{4} x_i^k \frac{\partial \psi_i^R}{\partial \xi} & \sum_{i=1}^{4} x_i^k \frac{\partial \psi_i^R}{\partial \eta} \\ \sum_{i=1}^{4} y_i^k \frac{\partial \psi_i^R}{\partial \xi} & \sum_{i=1}^{4} y_i^k \frac{\partial \psi_i^R}{\partial \eta} \end{pmatrix}$$

Taken into account the definition on the shape functions

$$J^T = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

- **Numerical Gaussian 2D quadrature**

$$\iint_K F(x,y)\,\mathrm{d}x\mathrm{d}y \approx \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j F(P(\xi_i,\eta_j), Q(\xi_i,\eta_j)) |J(\xi_i,\eta_j)|.$$

Quadrilateral Elements

Steps needed to integrate the function $F(x,y)$ in the quadrilateral Ω^k defined by $v_1,\ v_2,\ v_3,\ v_4$.

1. Choose N (Gauss degree formula) and take the Gaussian points (ξ_i,η_j) for the reference element
2. Compute $\mathbf{x}_{ij} = \begin{pmatrix} x_{ij} \\ y_{ij} \end{pmatrix} = \begin{pmatrix} P(\xi_i,\eta_j) \\ Q(\xi_i,\eta_j) \end{pmatrix}$ the Gaussian points in Ω_k
3. Compute the determinants $J_{ij} = |J(\xi_i,\eta_j)|$
4. Do the sum. For example N=2 we have

$$\iint_K F(x,y)\,\mathrm{d}x\mathrm{d}y \approx \omega_{11}F(\mathbf{x}_{11})J_{11} + \omega_{12}F(\mathbf{x}_{12})J_{12} + \omega_{21}F(\mathbf{x}_{21})J_{21} + \omega_{22}F(\mathbf{x}_{22})J_{22}$$

Quadrilateral Elements

- **Example:** Integrate the function $F(x,y) = x^2 - y^2$ in the quadrilateral Ω^k defined by

$v_1 = (0,0),\ v_2 = (5,-1),\ v_3 = (4,5),\ v_4 = (1,4).$

We will use the Gaussian quadrature formula: (N=2)

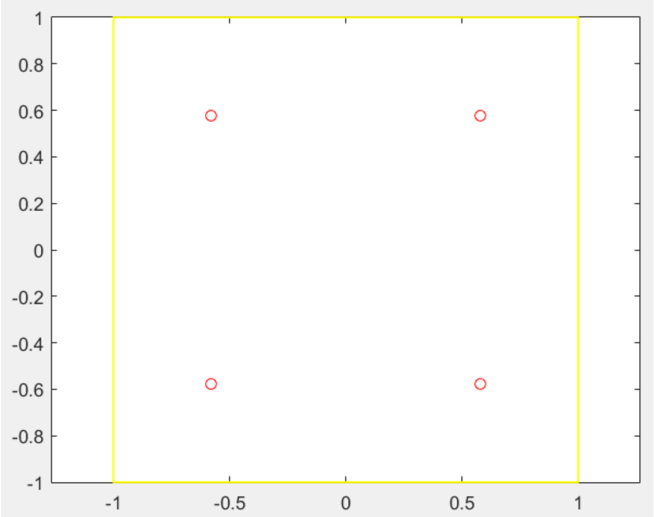
$$\iint_K F(x,y)\,\mathrm{d}x\mathrm{d}y \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j F(P(\xi_i,\eta_j), Q(\xi_i,\eta_j))|J(\xi_i,\eta_j)|.$$

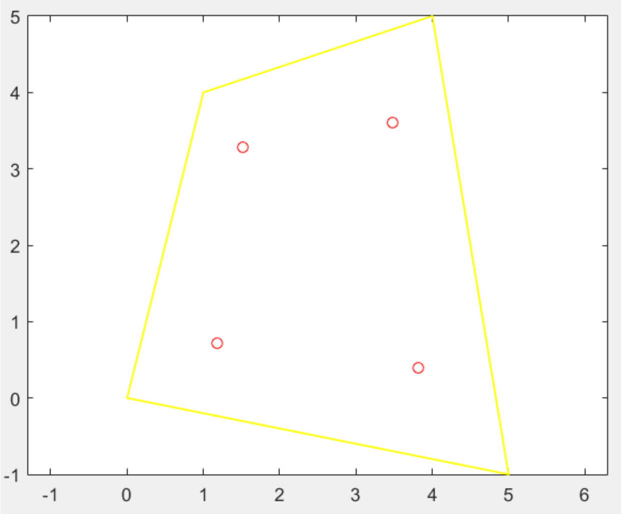
First we compute $x_{ij} = P(\xi_i,\eta_j)$, $y_{ij} = Q(\xi_i,\eta_j)$ where the (ξ_i,η_j) are the Gaussian points for the reference element.

$$\begin{pmatrix} x_{ij} \\ y_{ij} \end{pmatrix} = \begin{pmatrix} P(\xi_i,\eta_j) \\ Q(\xi_i,\eta_j) \end{pmatrix} = \begin{pmatrix} \psi_1^R(\xi,\eta)\,x_1^k + \psi_2^R(\xi,\eta)\,x_2^k + \psi_3^R(\xi,\eta)\,x_3^k + \psi_4^R(\xi,\eta)\,x_4^k \\ \psi_1^R(\xi,\eta)\,y_1^k + \psi_2^R(\xi,\eta)\,y_2^k + \psi_3^R(\xi,\eta)\,y_3^k + \psi_4^R(\xi,\eta)\,y_4^k \end{pmatrix}\Bigg|_{(\xi_i,\eta_j)}$$

Quadrilateral Elements

- For $N = 2$, the Gaussian points are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ we want to compute the corresponding points in our quadrilateral





Quadrilateral Elements

- For $N = 2$, the Gaussian points are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ and the shape functions are

$$\psi_1^R(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1-\eta)}{2}, \qquad \psi_2^R(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1-\eta)}{2},$$

$$\psi_3^R(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1+\eta)}{2}, \qquad \psi_4^R(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1+\eta)}{2}.$$

- Therefore, considering $v_1 = (0{,}0)$, $v_2 = (5,-1)$, $v_3 = (4{,}5)$, $v_4 = (1{,}4)$.

$$\begin{pmatrix} x_{11} \\ y_{11} \end{pmatrix} = \begin{pmatrix} 0\cdot\psi_1^R\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + 5\cdot\psi_2^R\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + 4\cdot\psi_3^R\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + 1\cdot\psi_4^R\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \\ 0\cdot\psi_1^R\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) - 1\cdot\psi_2^R\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + 5\cdot\psi_3^R\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + 4\cdot\psi_4^R\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \end{pmatrix} = \begin{pmatrix} 3.4880 \\ 3.6100 \end{pmatrix}$$

$$\begin{pmatrix} x_{12} \\ y_{12} \end{pmatrix} = \begin{pmatrix} 0\cdot\psi_1^R\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) + 5\cdot\psi_2^R\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) + 4\cdot\psi_3^R\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) + 1\cdot\psi_4^R\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) \\ 0\cdot\psi_1^R\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) - 1\cdot\psi_2^R\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) + 5\cdot\psi_3^R\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) + 4\cdot\psi_4^R\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) \end{pmatrix} = \begin{pmatrix} 3.8214 \\ 0.3900 \end{pmatrix}$$

$$\begin{pmatrix} x_{21} \\ y_{21} \end{pmatrix} = \begin{pmatrix} 1.5120 \\ 3.2767 \end{pmatrix}, \qquad \begin{pmatrix} x_{22} \\ y_{22} \end{pmatrix} = \begin{pmatrix} 1.1786 \\ 0.7233 \end{pmatrix}.$$

Quadrilateral Elements

- The Jacobian values are:

$$J^T = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 5 & -1 \\ 4 & 5 \\ 1 & 4 \end{bmatrix}$$

- Substituting the Gaussian points:

$$J_{11} = \det\left(J\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right) = 4.8557$$

$$J_{12} = \det\left(J\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\right) = 6.2990$$

$$J_{21} = \det\left(J\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right) = 3.7010$$

$$J_{22} = \det\left(J\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\right) = 5.1443$$

Quadrilateral Elements

- Finally using the formula for N=2:

$$\iint_K F(x,y)\,\mathrm{d}x\mathrm{d}y \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j F(P(\xi_i,\eta_j), Q(\xi_i,\eta_j))|J(\xi_i,\eta_j)|.$$

- We get:

$$\iint_K F(x,y)\,\mathrm{d}x\mathrm{d}y \approx$$

$$\omega_{11}F(x_{11},y_{11})J_{11} + \omega_{12}F(x_{12},y_{12})J_{12} + \omega_{21}F(x_{21},y_{21})J_{21} + \omega_{22}F(x_{22},y_{22})J_{22} =$$

$$1\cdot(-0.8660)\cdot 4.8557 + 1\cdot(14.4508)\cdot 6.2990 + 1\cdot(-8.4508)\cdot 3.7010 + 1\cdot(0.8660)\cdot 5.1443 =$$

$$= 60$$

Exercise: Integrate the function $\mathrm{F}(x,y) = x^2y^2$ in the domain [0,8]x[0,2] (**sol**: 455.11)

Quadrilateral Elements: Shape Functions

- **Shape Functions and 2D Gaussian quadrature:**

One important property about shape functions is that

$$\psi_i^k(x,y) = (\psi_i^k \circ \phi)(\xi,\eta) = \psi_i^R(\xi,\eta)$$

This simplify the integral computations when shape functions are involved (as it will be always for FME).

That means :

$$\iint_{\Omega_k} \psi_i^k(x,y)dxdy = \iint_R \psi_i^R(\xi,\eta) \cdot |J(\xi,\eta)| \cdot d\xi d\eta$$

Therefore, you need to compute the Jacobian, but no to compute the change of variables of the function.

Quadrilateral Elements: Shape Functions

Example: If Ω_k is the quadrilateral defined by the r vertices $v_1 = (0{,}0)$, $v_2 = (5,-1)$, $v_3 = (4{,}5)$, $v_4 = (1{,}4)$, the integral of the product of two shape functions is

$$S_{ij} = \iint_{\Omega_k} \psi_i^k(x,y) \cdot \psi_j^k(x,y) dxdy = \iint_R \psi_i^R(\xi,\eta) \cdot \psi_j^R(\xi,\eta) \cdot |J(\xi,\eta)| \cdot d\xi d\eta$$

If you compute all the possible products you will obtain:

$$S = \begin{bmatrix} 2.27780 & 1.25000 & 0.55556 & 1.00000 \\ 1.25000 & 2.72220 & 1.22220 & 0.55556 \\ 0.55556 & 1.22220 & 2.16670 & 0.97222 \\ 1.00000 & 0.55556 & 0.97222 & 1.72222 \end{bmatrix}$$

Quadrilateral Elements: Shape Functions

- **Derivative of the Shape Functions:**

In general during FEM formulation, not only integration of the shape functions are needed, but also the integrals of the shape function derivatives are involved. Let's consider the first derivative: (chain rule)

$$\frac{\partial \psi_i^k}{\partial x} = \frac{\partial \psi_i^k}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_i^k}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \psi_i^R}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_i^R}{\partial \eta}\frac{\partial \eta}{\partial x}.$$

Here we need to evaluate the inverse derivatives $\frac{\partial \xi}{\partial x}$ and $\frac{\partial \eta}{\partial x}$. For that we use the inverse function theorem:

$$\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}^{-1} = J^{-1}$$

Quadrilateral Elements: Shape Functions

- Therefore we need to compute:

$$J^T = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1-\eta) \\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

and invert these 2x2 matrix. In these case we can use Matlab or **explicitly**

If $J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$ then $J^{-1} = \frac{1}{J_{11}J_{22} - J_{12}J_{21}}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \equiv \begin{bmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix}$

Then

$$\iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y)dxdy = \iint_R \left(\frac{\partial \psi_i^R}{\partial \xi}(\xi,\eta)\tilde{J}_{11} + \frac{\partial \psi_i^R}{\partial \eta}(\xi,\eta)\tilde{J}_{21} \right) \cdot |J(\xi,\eta)| \cdot d\xi d\eta$$

Analogously

$$\iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial y}(x,y)dxdy = \iint_R \left(\frac{\partial \psi_i^R}{\partial \xi}(\xi,\eta)\tilde{J}_{12} + \frac{\partial \psi_i^R}{\partial \eta}(\xi,\eta)\tilde{J}_{22} \right) \cdot |J(\xi,\eta)| \cdot d\xi d\eta$$

Quadrilateral Elements: Shape Functions

- **Example**: If Ω_k is the quadrilateral defined by the vertices $v_1 = (0,0)$, $v_2 = (5,-1)$, $v_3 = (4,5)$, $v_4 = (1,4)$, the integral of the product of **two shape function derivatives** is

- $\iint_{\Omega_k} \frac{\partial \psi_1^k}{\partial x}(x,y) \frac{\partial \psi_1^k}{\partial x}(x,y) dxdy =$

$$\iint_R \left(\frac{\partial \psi_1^R}{\partial \xi}(\xi,\eta)\tilde{J}_{11} + \frac{\partial \psi_1^R}{\partial \eta}(\xi,\eta)\tilde{J}_{21} \right) \cdot \left(\frac{\partial \psi_1^R}{\partial \xi}(\xi,\eta)\tilde{J}_{11} + \frac{\partial \psi_1^R}{\partial \eta}(\xi,\eta)\tilde{J}_{21} \right) \cdot |J(\xi,\eta)| \cdot d\xi d\eta = 0.40995$$

If you compute all the possible products you will obtain:

$$S^{11} = \begin{bmatrix} 0.40995 & -0.36892 & -0.20479 & 0.16376 \\ -0.36892 & 0.34516 & 0.25014 & -0.22639 \\ -0.20479 & 0.25014 & 0.43155 & -0.47690 \\ 0.16376 & -0.22639 & -0.47690 & 0.53953 \end{bmatrix}$$

Where $S_{ij}^{11} = \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial x}(x,y) dxdy$