UNIVERSITAT POLITÈCNICA DE CATALUNYA
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Escola Tècnica Superior d'Enginyeria
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ETSEIB

Mètodes Numèrics:

A First Course on Finite Elements

1D Finite Elements: Examples

Following: *Curs d'Elements Finits amb Aplicacions* (J. Masdemont)
http://hdl.handle.net/2099.3/36166

Dept. Matemàtiques ETSEIB - UPC BarcelonaTech

Linear Elasticity 1D

Linear Elasticity 1D equation:

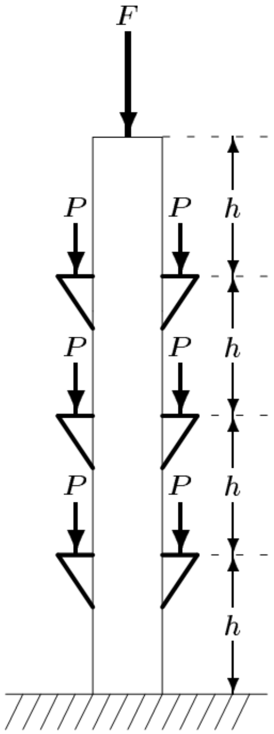
$$-\frac{d}{dx}\left(E(x)A(x)\frac{du(x)}{dx}\right)=0,$$

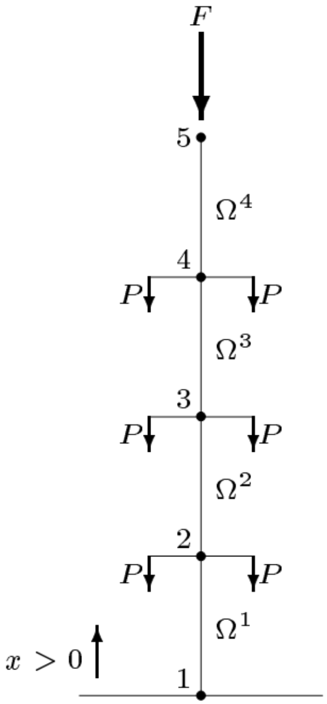
with $E(x)$ the material elasticity function (**Young modulus**), $A(x)$ the **section area** and $u(x)$ the **displacement**.

Linear Elasticity 1D

- **Example 1:** Constant loaded column

Let's assume *E·A* constant (homogeneous column).





$h = 4.5\,m$
$P = 11 \times 10^4 N$
$F = 3 \times 10^5 N$
$E = 2.0 \times 10^{11} N/m^2$
$A = 250\ cm^2$

Linear Elasticity 1D

- This is a particular case of the model equation

$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

with $a_1 = EA$ constant, $a_0 \equiv 0$ i $f(x) \equiv 0$ (if the column weight is not consider).

as we learned before, the problem can be stated as

$$[K^k]u^k = F^k + Q^k$$

$$[K^k] = [K^{k,1}] = \frac{EA}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad F^k = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{per} \quad k = 1, 2, 3, 4.$$

Linear Elasticity 1D

After assembly the system we obtain

$$\frac{EA}{h}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{pmatrix}$$

Finally, we impose

$$U_1 = 0\,m, \quad Q_2^1 + Q_1^2 = Q_2^2 + Q_1^3 = Q_2^3 + Q_1^4 = -2.2 \times 10^5\,N, \quad Q_2^4 = -3 \times 10^5\,N.$$

$$1.11 \times 10^9 \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = -\begin{pmatrix} 2.2 \\ 2.2 \\ 2.2 \\ 3.0 \end{pmatrix} \times 10^5$$

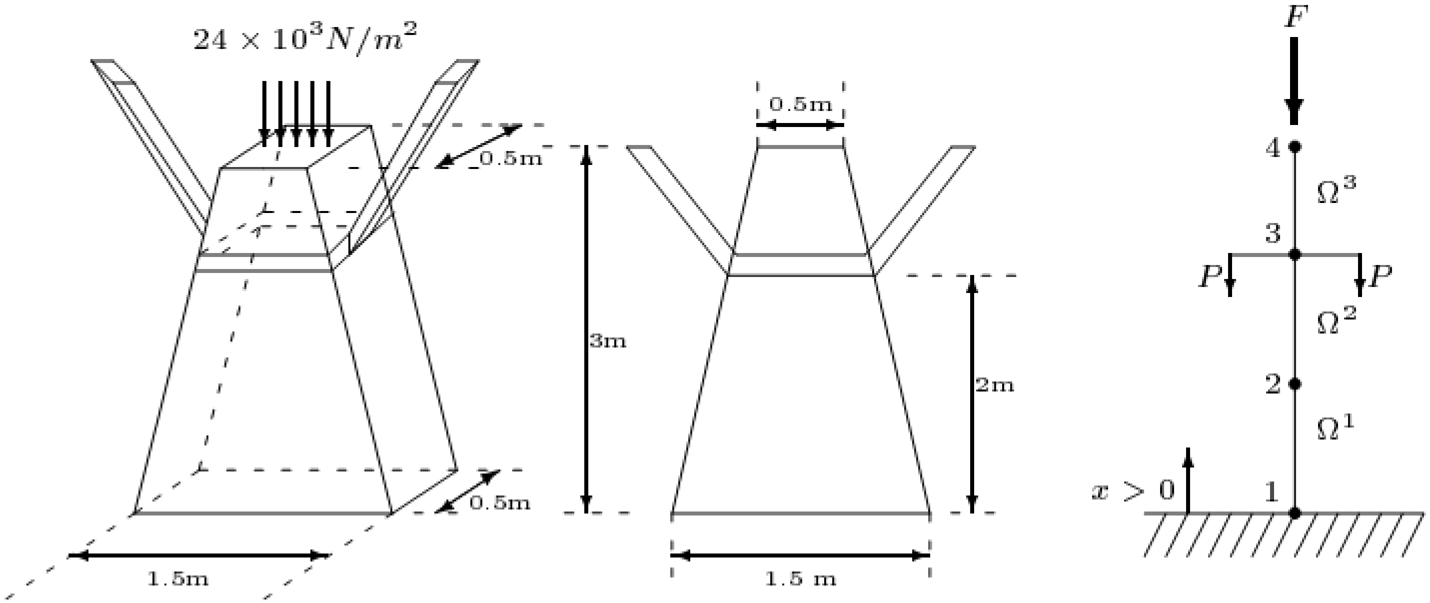
Solution:

$$U_1 = 0\,m, \quad U_2 = -0.86\,mm, \quad U_3 = -1.53\,mm, \quad U_4 = -2.00\,mm, \quad U_5 = -2.27\,mm.$$

$$Q_1^1 = \frac{EA}{h}(U_1 - U_2) = 9.6 \times 10^5\,N$$ (Reaction force on the ground)

Linear Elasticity 1D

Example 2: Concrete Pyramidal Column



Now the inner weight of the column is taken into account.

Consider $P = 2 \times 10^3\, N$ $\quad h = 1\, m$

$E = 28 \times 10^9\, N/m^2$ $\quad w = 25 \times 10^3\, N/m^3$ (specific concrete weight)

Linear Elasticity 1D

Now, the section area is not constant

$$A(x) = 0.5\left(1.5\,\frac{3-x}{3} + 0.5\,\frac{x}{3}\right) = \frac{3}{4} - \frac{x}{6}\,m^2.$$

and therefore, the **internal forces** $f(x) = \omega \frac{dV}{dx}$ for unit length can be expressed by the product

$$f(x) = -w\,A(x) = \left(\frac{25x}{6} - \frac{75}{4}\right) \times 10^3\,N/m,$$

(*Hint*: This can be obtained by the derivative of the formula for the volume of column above a level x.

$$V(x) = \frac{1}{2}\big(A(3) + A(x)\big) \cdot h$$

)

Linear Elasticity 1D

- Here, to obtain the linear system $[K^k]u^k = F^k + Q^k$

we need to compute

$$K_{ij}^k = K_{ij}^{k,1} = E\int_{x_A}^{x_B} A(x)\,\frac{d\psi_i^k}{dx}(x)\frac{d\psi_j^k}{dx}(x)\,dx, \qquad F_i^k = \int_{x_A}^{x_B} f(x)\,\psi_i^k(x)\,dx.$$

That gives **different values** for each element. Considering the first one, $\Omega^1 = [0,1]$, we obtain:

$$K_{11}^1 = K_{22}^1 = E\int_0^1 A(x)\,dx = \frac{2E}{3}, \qquad K_{12}^1 = K_{21}^1 = -E\int_0^1 A(x)\,dx = -\frac{2E}{3},$$

$$F_1^1 = -w\int_0^1 A(x)(1-x)\,dx = -\frac{25\,w}{72}, \qquad F_2^1 = -w\int_0^1 xA(x)\,dx = -\frac{23\,w}{72}.$$

Linear Elasticity 1D

- For the three elements we have:

$$[K^1] = \frac{2E}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad [K^2] = \frac{E}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad [K^3] = \frac{E}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$F^1 = -\frac{w}{72}\begin{pmatrix} 25 \\ 23 \end{pmatrix}, \qquad F^2 = -\frac{w}{72}\begin{pmatrix} 19 \\ 17 \end{pmatrix}, \qquad F^3 = -\frac{w}{72}\begin{pmatrix} 13 \\ 11 \end{pmatrix}.$$

and next, assembling the system

$$\frac{E}{6}\begin{pmatrix} 4 & -4 & 0 & 0 \\ -4 & 7 & -3 & 0 \\ 0 & -3 & 5 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = -\frac{w}{72}\begin{pmatrix} 25 \\ 42 \\ 30 \\ 11 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}$$

With BC

$$U_1 = 0, \qquad Q_2 = Q_2^1 + Q_1^2 = 0, \qquad Q_3 = Q_2^2 + Q_1^3 = -2P, \qquad Q_4 = Q_2^3 + Q_1^4 = -F,$$

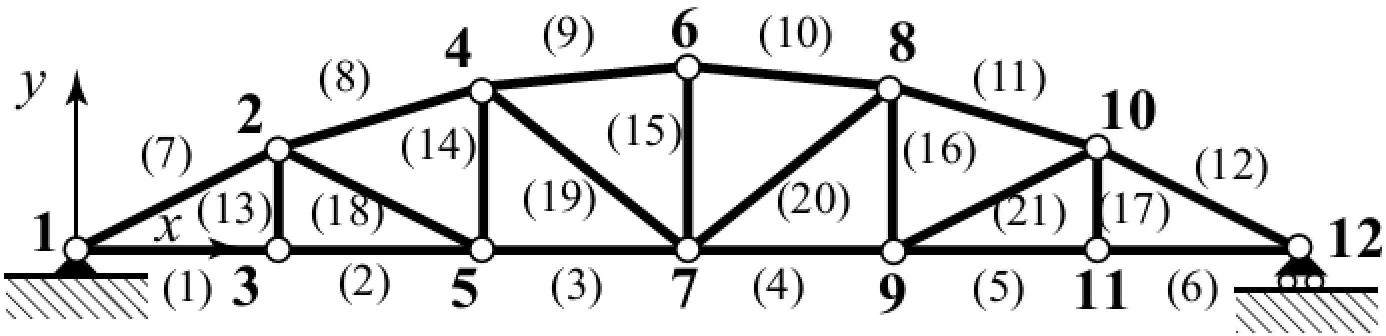
$$F = (0.5)^2 \times 24 \times 10^3 = 6000\,N.$$

Solution:

$$U_1 = 0\,m, \qquad U_2 = -2.08 \times 10^{-6}\,m, \qquad U_3 = -3.81 \times 10^{-6}\,m, \qquad U_4 = -4.86 \times 10^{-6}\,m.$$

Linear 1D elements on the Plane

When linear elements are part of a 2D structure:



Then, each node has **two degree of freedom** for their displacements, that is $u^i = (u_x^i, u_y^i)$. Therefore, the **size** of the final system of equations is now **2·Npoints**

For each element we have 4 degrees of freedom $\boldsymbol{u}^e = \begin{bmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \end{bmatrix}$

Linear 1D elements on the Plane (Truss Stiffness)

Adding a new coordinate, the 1D structural problem is now written as:

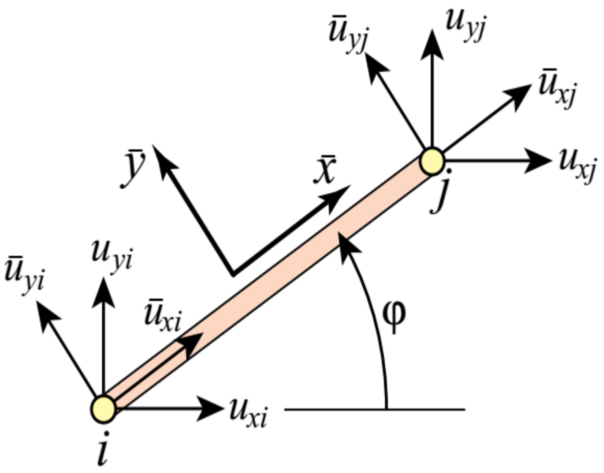
$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}$$

The stiffness matrix is then

$$\boxed{\bar{\mathbf{K}}^e = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.}$$

Linear 1D elements on the Plane

- The change in displacements is computed from a 2D rotation:



$$\bar{u}_{xi} = u_{xi}c + u_{yi}s, \qquad \bar{u}_{yi} = -u_{xi}s + u_{yi}c,$$
$$\bar{u}_{xj} = u_{xj}c + u_{yj}s, \qquad \bar{u}_{yj} = -u_{xj}s + u_{yj}c,$$

using $c = \cos\varphi$ and $s = \sin\varphi$

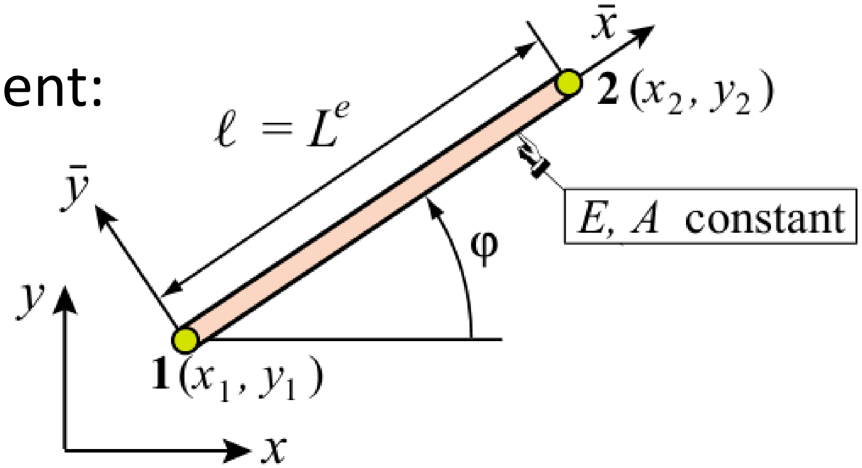
Or in matrix form

$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}}_{\mathbf{T}^e} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

Linear 1D elements on the Plane

Finally for each **structural 1D** element:

$$\boxed{\mathbf{K}^e = (\mathbf{T}^e)^T\, \bar{\mathbf{K}}^e\, \mathbf{T}^e.}$$



The element stiffness matrix is

$$\mathbf{K}^e = \frac{EA}{\ell}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

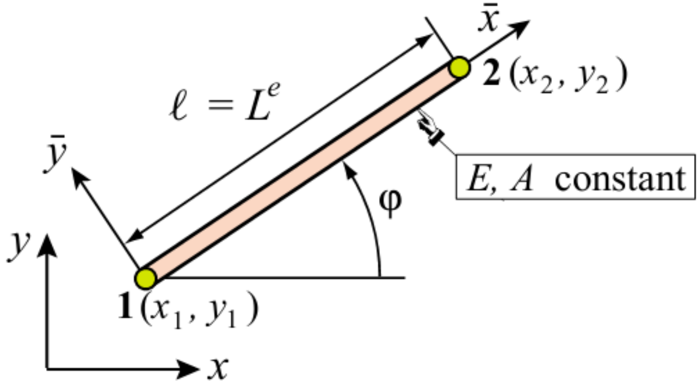
using

$$c = \cos\varphi$$

$$s = \sin\varphi$$

Linear 1D elements on the Plane

It can be expressed in terms of the node coordinates as:



$$\mathbf{K}^e = \frac{EA}{\ell^3}\begin{bmatrix} x_{21}x_{21} & x_{21}y_{21} & -x_{21}x_{21} & -x_{21}y_{21} \\ x_{21}y_{21} & y_{21}y_{21} & -x_{21}y_{21} & -y_{21}y_{21} \\ -x_{21}x_{21} & -x_{21}y_{21} & x_{21}x_{21} & x_{21}y_{21} \\ -x_{21}y_{21} & -y_{21}y_{21} & x_{21}y_{21} & y_{21}y_{21} \end{bmatrix}$$

where $c = \cos\varphi = x_{21}/\ell, s = \sin\varphi = y_{21}/\ell$

$$x_{21} = x_2 - x_1, y_{21} = y_2 - y_1, \ell = \sqrt{x_{21}^2 + y_{21}^2}$$

Linear 1D elements on the Plane

The **final assembled system** is obtained in the same way as in the 1D case. The only difference is the final size, it is **double** of the usual one, because the unknown essential variable vector (displacements) is now:

$$\boldsymbol{u} = (u_x^1, u_y^1, u_x^2, u_y^2, \ldots\ldots, u_x^N, u_y^N)'$$

The assembled force vector have also the same size

$$\boldsymbol{f} = (f_x^1, f_y^1, f_x^2, f_y^2, \ldots\ldots, f_x^N, f_y^N)'$$

And the system

$$\boldsymbol{K} \cdot \boldsymbol{u} = \boldsymbol{f}$$

(examples in the Matlab sessions)

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2D Finite Elements Applications

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Poisson's Equation

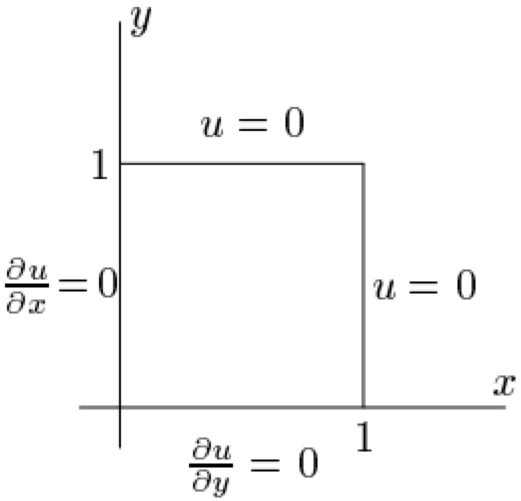
- Let's consider the **Poisson's equation** on a rectangle:

$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=1 \quad \text{on} \quad \Omega=[0,1]\times[0,1],$$

With BC

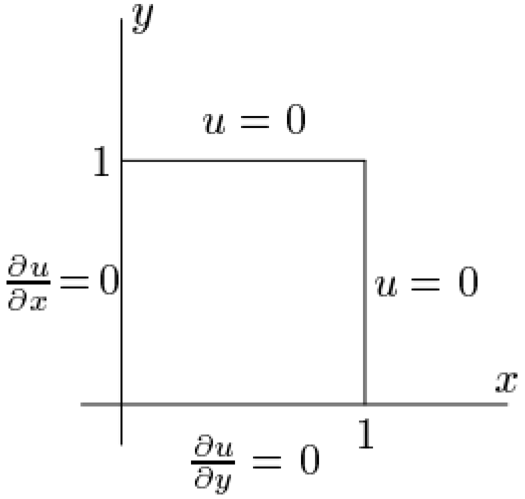
$$u(x,1) = \frac{\partial u}{\partial y}(x,0) \equiv 0 \qquad 0\le x\le 1,$$

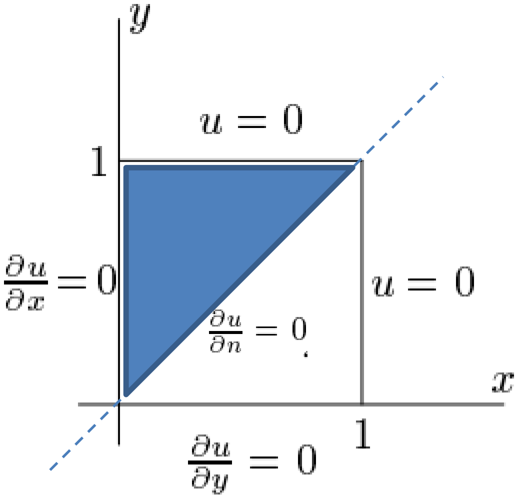
$$u(1,y)=\frac{\partial u}{\partial x}(0,y)\equiv 0, \qquad 0\le y\le 1,$$



Poisson's Equation

- If there is a **symmetry in the problem**, we can use it to study only half of the domain. In this case, the diagonal is now a new boundary and its BC will be always $$\frac{\partial u}{\partial n} \equiv \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \equiv 0,$$



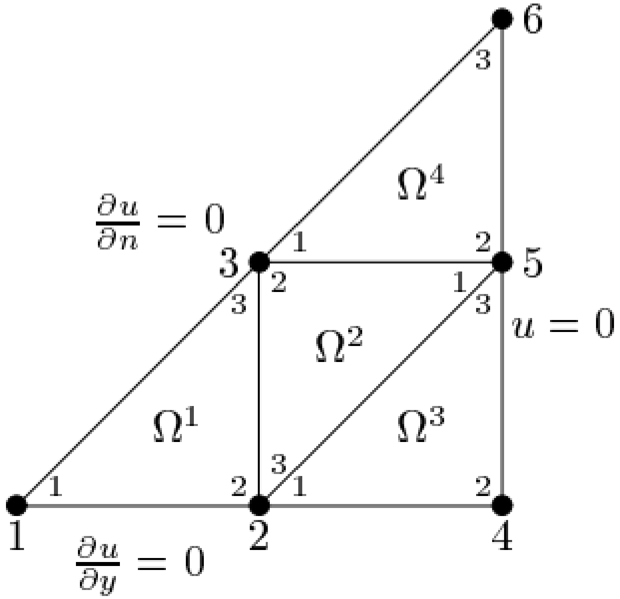


Poisson's Equation

- Then, we can consider for example, the triangular elements defined in the figure.

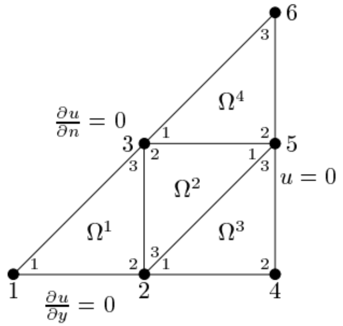
Connectivity matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$



Hint: Notice that the local enumeration must be counter-clockwise in order to preserve orientation

Poisson's Equation



- Assembling the equations:

$$\begin{pmatrix} K^1_{11} & K^1_{12} & K^1_{13} & 0 & 0 & 0 \\ K^1_{21} & K^1_{22}+K^2_{33}+K^3_{11} & K^1_{23}+K^2_{32} & K^3_{12} & K^2_{31}+K^3_{13} & 0 \\ K^1_{31} & K^1_{32}+K^2_{23} & K^1_{33}+K^2_{22}+K^4_{11} & 0 & K^2_{21}+K^4_{12} & K^4_{13} \\ 0 & K^3_{21} & 0 & K^3_{22} & K^3_{23} & 0 \\ 0 & K^2_{13}+K^3_{31} & K^2_{12}+K^4_{21} & K^3_{32} & K^2_{11}+K^3_{33}+K^4_{22} & K^4_{23} \\ 0 & 0 & K^4_{31} & 0 & K^4_{32} & K^4_{33} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{pmatrix} =$$

$$= \begin{pmatrix} F^1_1 \\ F^1_2+F^2_3+F^3_1 \\ F^1_3+F^2_2+F^4_1 \\ F^3_2 \\ F^2_1+F^3_3+F^4_2 \\ F^4_3 \end{pmatrix} + \begin{pmatrix} Q^1_1 \\ Q^1_2+Q^2_3+Q^3_1 \\ Q^1_3+Q^2_2+Q^4_1 \\ Q^3_2 \\ Q^2_1+Q^3_3+Q^4_2 \\ Q^4_3 \end{pmatrix}$$

Poisson’s Equation

- For the Poisson’s equation, for a general linear triangle we have

$$K_{ij}^k = \frac{1}{4A_k}(a_{11}^k\beta_i^k\beta_j^k + a_{22}^k\gamma_i^k\gamma_j^k) \underset{a_{11}^k=a_{22}^k=1,\ \forall k}{\underset{\uparrow}{=}} \frac{1}{4A_k}(\beta_i^k\beta_j^k+\gamma_i^k\gamma_j^k),$$

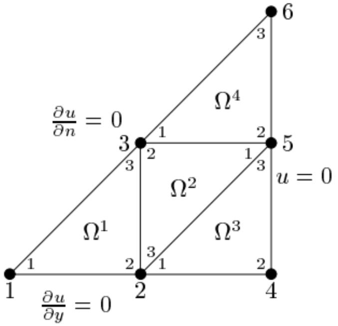
If the vertices of the triangle are $v_i = (x_i, y_i)$ we define:

$$\beta_i = y_j - y_k$$
$$\gamma_i = -(x_j - x_k)$$

$(i, j, k) = (1,2,3)$ and cyclic permutation

- If we take the **first element** Ω_1, with vertices

$$(x_1, y_1) = (0,0),\ (x_2, y_2) = (\tfrac{1}{2}, 0),\ (x_3, y_3) = (\tfrac{1}{2}, \tfrac{1}{2}) \qquad A_1 = \tfrac{1}{8}.$$



Then we get

$$[K^1] = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix}$$

In this case, all the triangles are rectangular (node 2 at angle of 90°) and with the same Area, so $[K^2] = [K^3] = [K^4] = [K^1].$

Poisson's Equation

- Compute now the F vector with $f = f_0 = 1$

We know $F_i^k = \int_{\Omega^k} f\,\psi_i\,dx\,dy = \frac{1}{3} f_0 A_k$

So

$$F_i^k = \frac{1}{3} f_0 \frac{1}{8} = \frac{f_0}{24}, \quad F^1 = F^2 = F^3 = F^4 = \frac{f_0}{24}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus, for each element $\Omega^k,\ k = 1 \ldots 4$ we get

$$\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} u_1^k \\ u_2^k \\ u_3^k \end{pmatrix} = \frac{f_0}{24}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} Q_1^k \\ Q_2^k \\ Q_3^k \end{pmatrix}$$

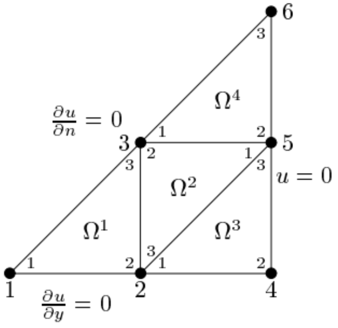
Poisson's Equation

- Using the connectivity matrix we can assemble the system and get:

$$\frac{1}{2}\left(\begin{array}{rrr|rrr} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ \hline 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{pmatrix} = \frac{f_0}{24}\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} Q_1^1 \\ Q_2^1 + Q_3^2 + Q_1^3 \\ Q_3^1 + Q_2^2 + Q_1^4 \\ Q_2^3 \\ Q_1^2 + Q_3^3 + Q_2^4 \\ Q_3^4 \end{pmatrix}$$

We'll see that BC will imply

$$U_4 = U_5 = U_6 = 0$$
$$Q_1 = Q_2 = Q_3 = 0.$$

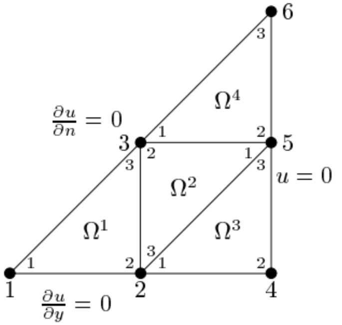


where

$$Q_1 = Q_1^1, \qquad Q_4 = Q_2^3,$$
$$Q_2 = Q_2^1 + Q_3^2 + Q_1^3, \qquad Q_5 = Q_1^2 + Q_3^3 + Q_2^4,$$
$$Q_3 = Q_3^1 + Q_2^2 + Q_1^4, \qquad Q_6 = Q_3^4.$$

Notice that nodes 4 and 6 have two BC, in these cases, **essential** ones are respected

Poisson's Equation



- Let's consider node 3, we get $Q_3 = Q^1_3 + Q^2_2 + Q^4_1.$

then, in general:

$$Q_3 = (Q^1_{31} + Q^1_{32} + Q^1_{33}) + (Q^2_{21} + Q^2_{22} + Q^2_{23}) + (Q^4_{11} + Q^4_{12} + Q^4_{13}),$$

From the integrals on the edges we have $Q^1_{31} = Q^2_{23} = Q^4_{12} = 0$

and from the balance of interior edges we have $Q^1_{32} + Q^2_{22} = 0,$ $Q^2_{21} + Q^4_{11} = 0,$

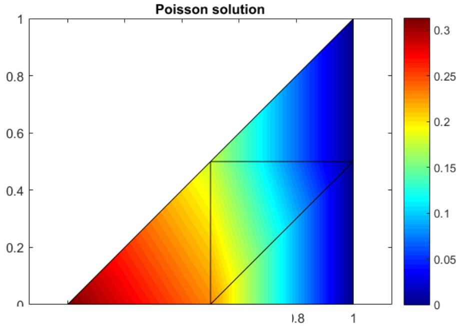
In total we get $Q_3 = Q^1_{33} + Q^4_{13}.$

What says that only the edges on the boundary of the domain are significant. Now,

$$Q_3 = Q^1_{33} + Q^4_{13} = \int_{\Gamma^1_3} q^1_{n3}(s)\psi^1_{33}(s)\, ds + \int_{\Gamma^4_3} q^4_{n3}(s)\psi^4_{13}(s)\, ds = 0,$$

Because $q^1_{n3} = q^4_{n3} = n_x \dfrac{\partial u}{\partial x} + n_y \dfrac{\partial u}{\partial y} =: \dfrac{\partial u}{\partial \vec{n}} \equiv 0,$

Poisson's Equation



- After applying all the BC we get

$$\frac{1}{2}\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ \hline 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{f_0}{24}\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ Q_4 \\ Q_5 \\ Q_6 \end{pmatrix}$$

Using the three first equation we obtain $\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 0.31250 \\ 0.22917 \\ 0.17708 \end{pmatrix}$

and with the rest

$$\begin{pmatrix} Q_4 \\ Q_5 \\ Q_6 \end{pmatrix} = -\frac{1}{24}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & -0.5 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} -0.156250 \\ -0.302083 \\ -0.041667 \end{pmatrix}$$

Heat Transfer

The 2D thermal equation is

$$-\frac{\partial}{\partial x}\left(k_c\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(k_c\frac{\partial T}{\partial y}\right)=f(x,y)$$

$T = T(x,y)$ is the **temperature** at the point (x,y) (units $°C$)

$k_c = k_{c_x} = k_{c_y}$ (if *Isotropic*) is the **thermal conductivity** coefficient (units $\frac{W}{m\,°C}$)

$f(x,y)$ is only present if there is some **internal heat generation** (units $\frac{W}{m^3}$)

As you can see, in the case of isotropic materials, it corresponds to the Poisson's equation.

Boundary Conditions:

$T(x,y) = T_0$, **fixed** temperature. **Essential** BC

$\frac{\partial T}{\partial n}(x,y) = 0$, null heat flux (**isolated**) . **Natural** BC.

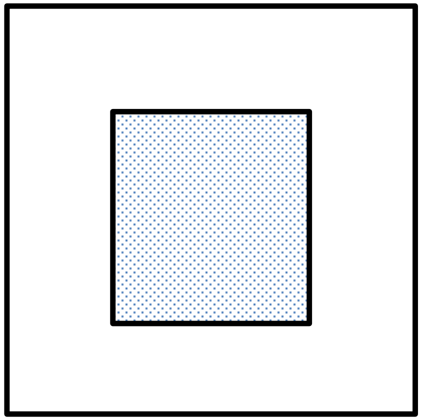
$k_c\frac{\partial T}{\partial n} + \beta(T - T_\infty) = 0$, **convection** along the Boundary. Mixed BC.

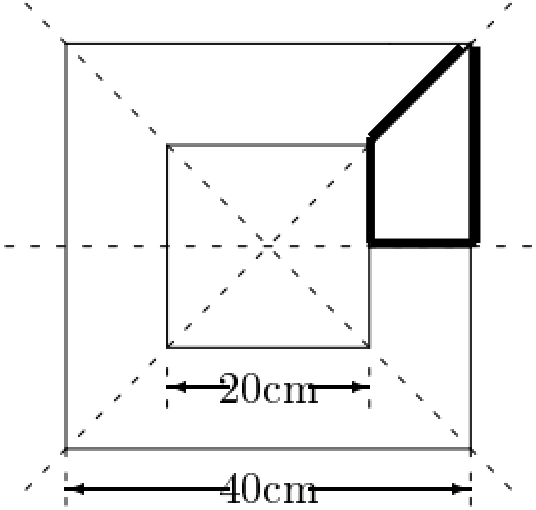
β convection coefficient

T_∞ bulk temperature

Heat Transfer

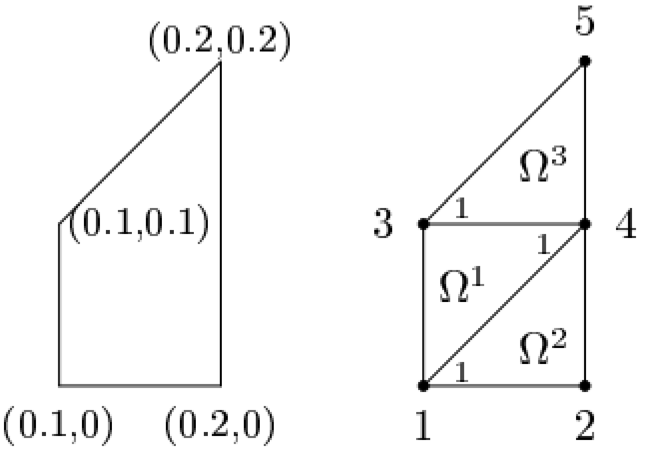
- Consider now the problem of computing the temperature distribution on a **square chimney** and consider its **plane section**. This problem is very symmetric and we can restrict the study to only a small piece of the total domain





Heat Transfer

- The problem can be restricted to the domain defined by:



We consider 3 elements and the equation (Poisson's constant coeff.)

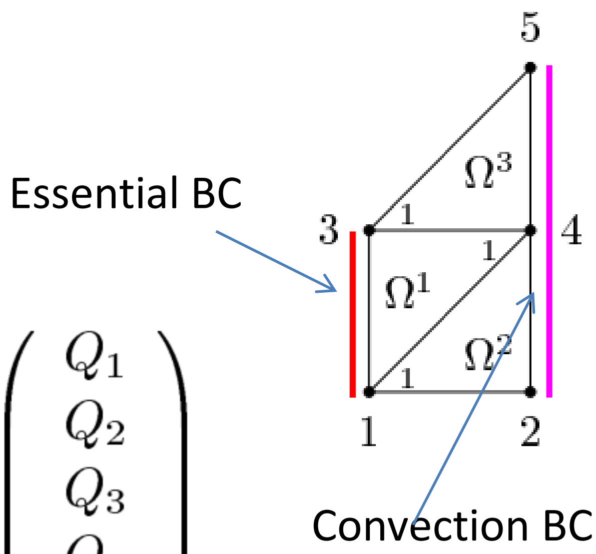
$$-\frac{\partial}{\partial x}\left(k_c\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(k_c\frac{\partial T}{\partial y}\right)=0$$

Therefore, for all three elements we get

$$[K^k]=\frac{1}{4A}\begin{pmatrix} 0.01 & -0.01 & 0.00 \\ -0.01 & 0.02 & -0.01 \\ 0.00 & -0.01 & 0.01 \end{pmatrix}, \qquad k=1,2,3,$$

$$A = 0.005\,m^2$$

Heat Transfer



Essential BC

Convection BC

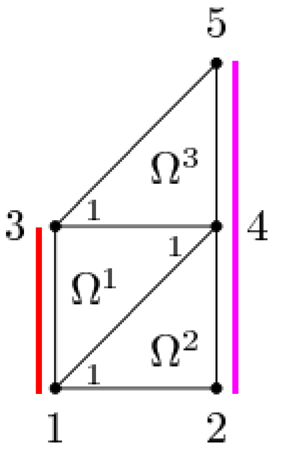
- The assembled system is

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 3 & -2 & 0 \\ 0 & -1 & -2 & 4 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix}$$

- Now impose the BC:

$T = 100°\mathrm{C}$ at nodes 1 and 3, that is: $T_1 = 100°\mathrm{C}, T_3 = 100°\mathrm{C}$
and **convection** on the other nodes 2, 4, 5 with
$\beta = 20$, convection coefficient
$T_\infty = 30$, bulk temperature

Heat Transfer



- The convection condition is

$$q_j^k(s) = -\beta(T_j^k(s) - T_\infty)$$

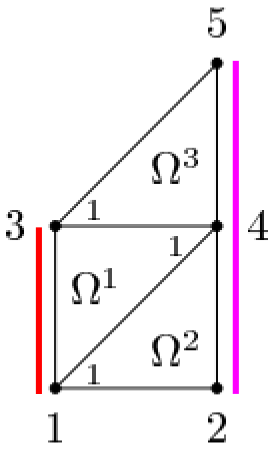
where *s* is the length parameter on the edges.

- If we proceed as usually, we will write

$$Q_2 = Q_{22}^2, \qquad Q_4 = Q_{32}^2 + Q_{22}^3, \qquad Q_5 = Q_{32}^3,$$

Because $Q_{21}^2 = 0$ and $Q_{33}^3 = 0$, due to the symmetry condition in those edges.

Heat Transfer



- Let's compute Q_{32}^2 as an example:

$$Q_{32}^2 = \int_{\Gamma_2^2} q_2^2(s)\psi_{32}^2(s)ds = -\beta\int_0^{h_2^2}(T_2^2(s)-T_\infty)\left(\frac{s}{h_2^2}\right)ds,$$

Here $h_2^2 = 0.1\,m.$ In the other side, at the second edge of Ω^2, function $T(x,y)$ is expressed as

$$T_2^2(s) = T_2^2\left(1-\frac{s}{h_2^2}\right) + T_3^2\left(\frac{s}{h_2^2}\right) = T_2\left(1-\frac{s}{h_2^2}\right) + T_4\left(\frac{s}{h_2^2}\right),$$

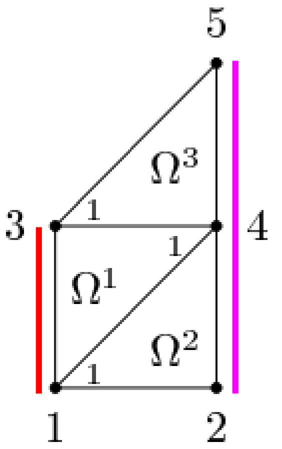
Therefore,

$$Q_{32}^2 = -\beta\int_0^{\frac{1}{10}}\left(T_2(1-10s)(10s) + T_4(10s)^2 - T_\infty(10s)\right)ds = -\beta\left(\frac{T_2}{60}+\frac{T_4}{30}-\frac{T_\infty}{20}\right)$$

Analogously

$$Q_{22}^2 = -\beta\left(\frac{T_2}{30}+\frac{T_4}{60}-\frac{T_\infty}{20}\right),\quad Q_{22}^3 = -\beta\left(\frac{T_4}{30}+\frac{T_5}{60}-\frac{T_\infty}{20}\right),\quad Q_{32}^3 = -\beta\left(\frac{T_4}{60}+\frac{T_5}{30}-\frac{T_\infty}{20}\right)$$

Heat Transfer



- That can written in matrix form as

$$\begin{pmatrix} Q_2 \\ Q_4 \\ Q_5 \end{pmatrix} = -\frac{\beta}{60}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} T_2 \\ T_4 \\ T_5 \end{pmatrix} + \frac{\beta T_\infty}{20}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Inserting now this in the assembled system we get,

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} T_2 \\ T_4 \\ T_5 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -100 \\ -200 \\ 0 \end{pmatrix} - \frac{\beta}{60}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} T_2 \\ T_4 \\ T_5 \end{pmatrix} + \frac{\beta T_\infty}{20}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

From essential BC From convection BC

$$\left(+ \frac{\beta}{60}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}\right)\begin{pmatrix} T_2 \\ T_4 \\ T_5 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -100 \\ -200 \\ 0 \end{pmatrix} + \frac{\beta T_\infty}{20}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Solution:

$$T_2 = 53.23\,^\circ C, \qquad T_4 = 52.32\,^\circ C, \qquad T_5 = 33.19\,^\circ C,$$

Heat Transfer

- **Matrix formulation** of convection BC

As we see the convection condition forces to **modify the global assembled matrix.** This can be expressed before assembly as an element matrix summation as follows:

$$\left([K^k]+[K^{k,c}]\right)T^k = F^k + F^{k,c} + Q^k$$

Where for **Linear Triangular elements** we have

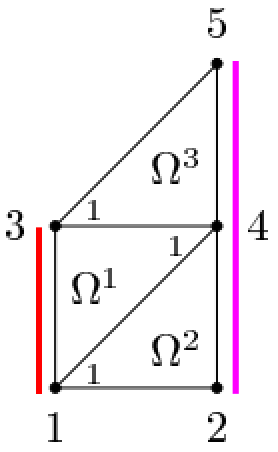
$$[K^{k,c}] = \frac{\bar{\beta}_1^k h_1^k}{6}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\bar{\beta}_2^k h_2^k}{6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \frac{\bar{\beta}_3^k h_3^k}{6}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$F^{k,c} = \frac{\bar{\beta}_1^k T_{\infty,1}\, h_1^k}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{\bar{\beta}_2^k T_{\infty,2}\, h_2^k}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{\bar{\beta}_3^k T_{\infty,3}\, h_3^k}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Only need to compute those terms if some of the edges of Ω^k are **convection edges**, otherwise they are just zero.

Exercise: Identify these matrices on the previous example

Heat Transfer



In our example, the convection condition says that elements Ω_2, Ω_3 have **edge number 2** (in both cases) affected by the convection BC. Thus, for *k*=2, 3 we have

$$\left([K^k]+[K^{k,c}]\right) T^k = F^k + F^{k,c} + Q^k$$

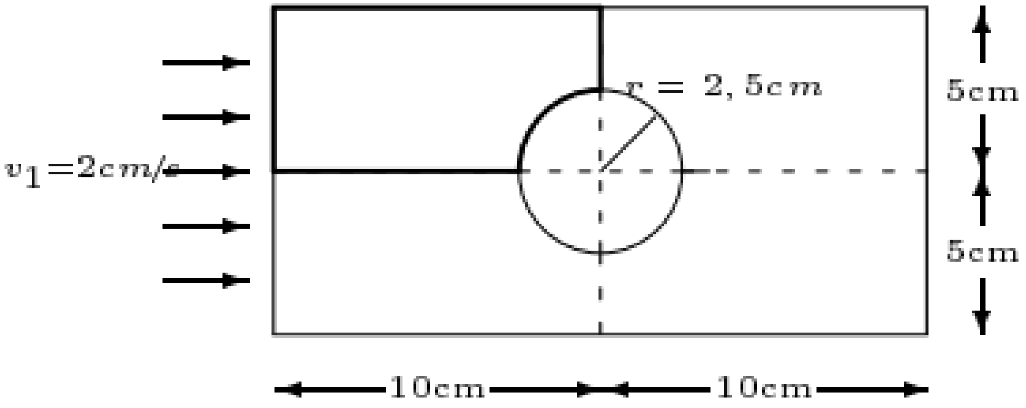
With

$$[K^{k,c}] = \frac{\bar{\beta}_2^k h_2^k}{6} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad F^{k,c} = \frac{\bar{\beta}_2^k T_{\infty,2} h_2^k}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

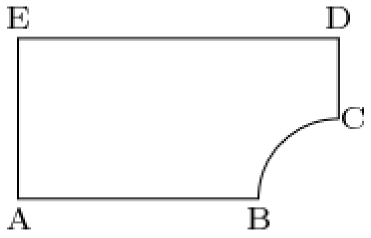
Where $\bar{\beta}_2^k = 20,\ T_\infty = 30$°C and $h_2^k = 0.1$

Fluids: flow around obstacles

- Consider the study of the velocity field of an ideal incompressible fluid passing through a pipeline with a cylindrical obstacle in the middle.



By symmetry:



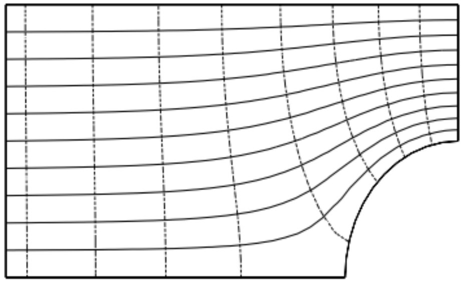
Fluid Flux around obstacles

- The equation of the problem is $\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=0$ (Poisson's Equation)

being $\Psi(x,y)$ the **stream function** of the fluid. That satisfy

$$\frac{\partial\Psi}{\partial y}=v_x, \qquad \frac{\partial\Psi}{\partial x}=-v_y$$

By definition, **Streamlines** are a family of curves that are instantaneously **tangent** to the **velocity vector** of the flow. These show the direction in which a massless fluid element will travel at any point in time. Different streamlines at the same instant in a flow **do not intersect**, because a fluid particle cannot have two different velocities at the same point.

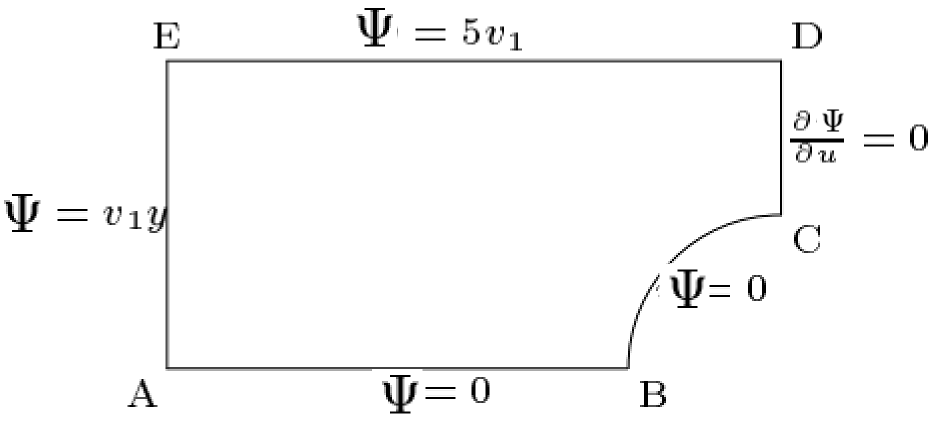


Stream lines

Fluid Flux around obstacles

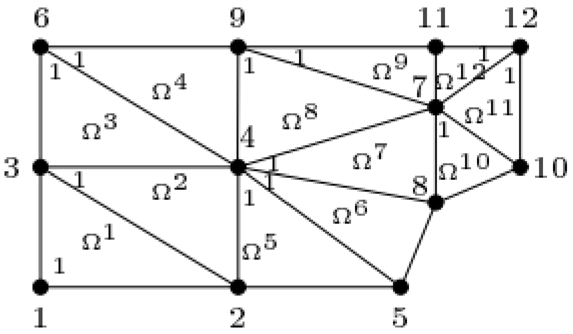
- To fix the **BC**, first we consider that the velocity field depends on the relative difference of two stream lines. In order to obtain a constant inflow velocity $v_x = v_1$ we need to impose $\frac{\partial \Psi}{\partial y} = v_1$ along the AE edge with $y \in [0,5]$. We obtain the following BC

$$v_1 = 2\, cm/s;$$



Fluid Flux around obstacles

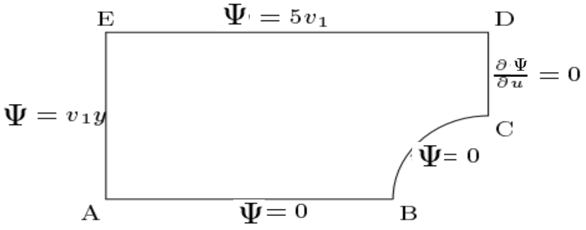
Define Elements:



with coordinates:

node	1	2	3	4	5	6	7	8	9	10	11	12
x	0.0	5.0	0.0	5.0	7.5	0.0	8.23	8.23	5.0	10.0	8.23	10.0
y	0.0	0.0	2.5	2.5	0.0	5.0	3.75	1.77	5.0	2.5	5.0	5.0

From the **BC** we know:



$$\Psi_1 = \Psi_2 = \Psi_5 = \Psi_8 = \Psi_{10} = 0$$

$$\Psi_6 = \Psi_9 = \Psi_{11} = \Psi_{12} = 10$$

$$\Psi_3 = 5$$

and from the balance at interior nodes $Q_4 = Q_7 = 0$

Fluids: flow around obstacles

- Using now the values of the Poisson's equation for triangles we have:

$$[K^1] = \begin{pmatrix} 1.25 & -0.25 & -1.00 \\ -0.25 & 0.25 & 0.00 \\ -1.00 & 0.00 & 1.00 \end{pmatrix}, \quad [K^2] = \begin{pmatrix} 0.25 & -0.00 & -0.25 \\ -0.00 & 1.00 & -1.00 \\ -0.25 & -1.00 & 1.25 \end{pmatrix}, \quad [K^3] = \begin{pmatrix} 1.00 & -1.00 & 0.00 \\ -1.00 & 1.25 & -0.25 \\ 0.00 & -0.25 & 0.25 \end{pmatrix},$$

$$[K^4] = \begin{pmatrix} 0.25 & -0.00 & -0.25 \\ -0.00 & 1.00 & -1.00 \\ -0.25 & -1.00 & 1.25 \end{pmatrix}, \quad [K^5] = \begin{pmatrix} 0.50 & -0.50 & 0.00 \\ -0.50 & 1.00 & -0.50 \\ 0.00 & -0.50 & 0.50 \end{pmatrix}, \quad [K^6] = \begin{pmatrix} 0.29 & -0.09 & -0.21 \\ -0.09 & 0.88 & -0.79 \\ -0.21 & -0.79 & 1.00 \end{pmatrix},$$

$$[K^7] = \begin{pmatrix} 0.31 & -0.19 & -0.11 \\ -0.19 & 0.94 & -0.74 \\ -0.11 & -0.74 & 0.86 \end{pmatrix}, \quad [K^8] = \begin{pmatrix} 0.74 & -0.55 & -0.19 \\ -0.55 & 0.74 & -0.19 \\ -0.19 & -0.19 & 0.39 \end{pmatrix}, \quad [K^9] = \begin{pmatrix} 0.19 & -0.00 & -0.19 \\ -0.00 & 1.29 & -1.29 \\ -0.19 & -1.29 & 1.49 \end{pmatrix},$$

$$[K^{10}] = \begin{pmatrix} 0.52 & -0.32 & -0.21 \\ -0.32 & 0.67 & -0.35 \\ -0.21 & -0.35 & 0.56 \end{pmatrix}, \quad [K^{11}] = \begin{pmatrix} 0.53 & -0.35 & -0.18 \\ -0.35 & 0.71 & -0.35 \\ -0.18 & -0.35 & 0.53 \end{pmatrix}, \quad [K^{12}] = \begin{pmatrix} 0.35 & -0.35 & 0.00 \\ -0.35 & 1.06 & -0.71 \\ 0.00 & -0.71 & 0.71 \end{pmatrix}.$$

We only need to assembly the equations for the 4 and 7 nodes

$$\begin{pmatrix} 0.00 & -1.50 & -0.50 & 4.34 & -0.09 & 0.00 & -0.31 & -0.40 & -1.55 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.31 & 0.00 & 0.00 & 4.47 & -1.06 & -0.19 & -0.56 & -2.00 & -0.35 \end{pmatrix}$$

and finally

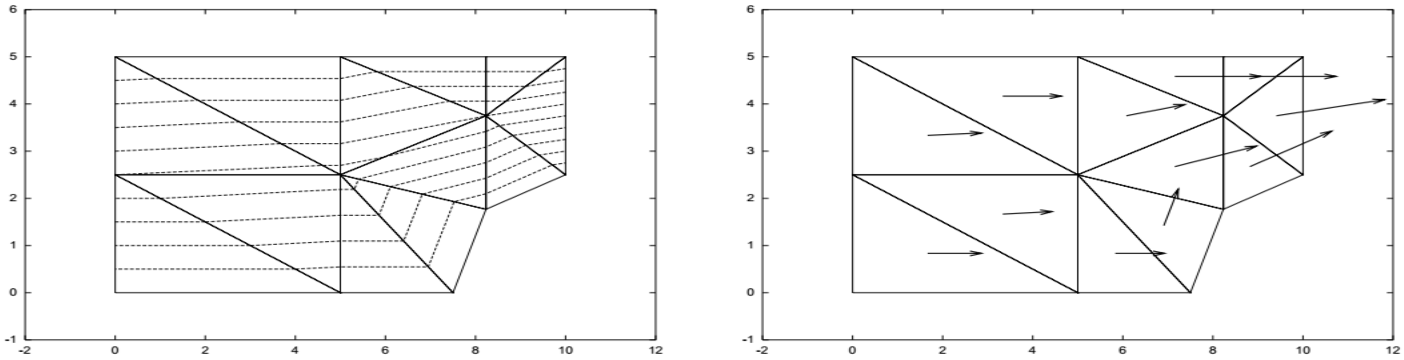
$$\begin{pmatrix} 4.34 & -0.31 \\ -0.31 & 4.47 \end{pmatrix} \begin{pmatrix} \Psi_4 \\ \Psi_7 \end{pmatrix} = \begin{pmatrix} 18.00 \\ 25.46 \end{pmatrix}, \qquad \Psi_4 = 4.56, \quad \Psi_7 = 6.00.$$

Fluids: flow around obstacles

- We can recover the values of the two components of the velocity in each element using interpolation

$$v_1^k = \frac{\partial \Psi}{\partial y} \simeq \sum_{i=1}^{3} \frac{\Psi_i^k \gamma_i^k}{2A_k}, \qquad v_2^k = -\frac{\partial \Psi}{\partial x} \simeq -\sum_{i=1}^{3} \frac{\Psi_i^k \beta_i^k}{2A_k}.$$

this values are plotted on the barycenter of each triangle on the following figure:



Fluids: flow around obstacles

- Using more elements you can compute a better approach