UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria
Industrial de Barcelona
ETSEIB

Mètodes Numèrics:

A First Course on Finite Elements

Finite Elements

Following: *Curs d'Elements Finits amb Aplicacions* (J. Masdemont)
http://hdl.handle.net/2099.3/36166

Dept. Matemàtiques ETSEIB - UPC BarcelonaTech

Finite Elements

Differential Equations (physical problem):

1-dim: Let's assume that $u(x)$ is a **magnitude** (temperature, displacement, etc.)

$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

1D Model Equation

Finite Elements

Differential Equations (physical problem):

1-dim: Let's assume that $u(x)$ is a **magnitude** (temperature, displacement, etc.)

$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

1D Model Equation

2-dim: Now $u(x, y)$ is a **magnitude** (temperature, etc.)

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}u = f,$$

2D Model Equation

Finite Elements

Differential Equations (physical problem):

1-dim: Let's assume that $u(x)$ is a **magnitude** (temperature, displacement, etc.)

$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

1D Model Equation

2on Order Terms

2-dim: Now $u(x,y)$ is a **magnitude** (temperature, etc.)

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}u = f,$$

2D Model Equation

Finite Elements

Procedure :

- First Step: To **set up and express** the equation at each element (element linear eq. system)
  (2x2 or 3x3)

- Second Step: To **assemble** the contribution of each element (global linear eq. system)
  (NxN)

- Third Step: To **solve** the linear eq. system

Finite Elements

?

Finite Elements

Stp1: Discretize in **elements** → **Meshing the domain**

Stp2: Write the **variational** equations

Stp3: Build the Linear System
Impose the BC

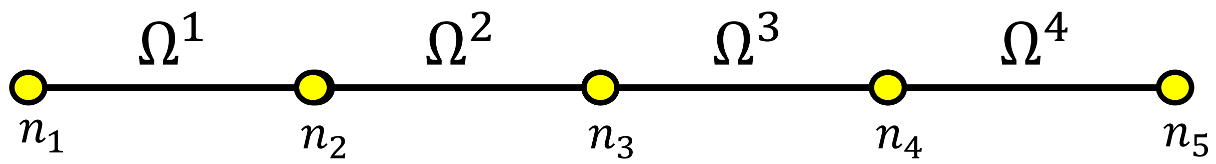
Stp4: Get **nodes** solution

Stp5: Extent solution to the Domain

Finite Elements

Element decomposition (meshing):

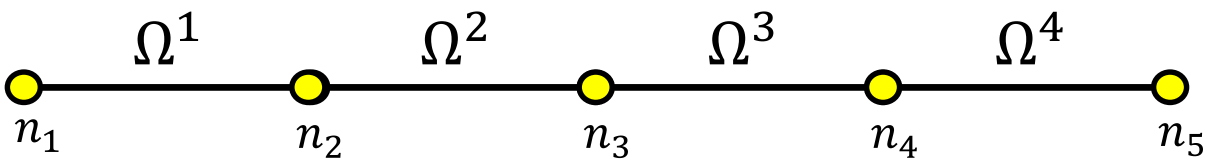
1-dim: (a line)



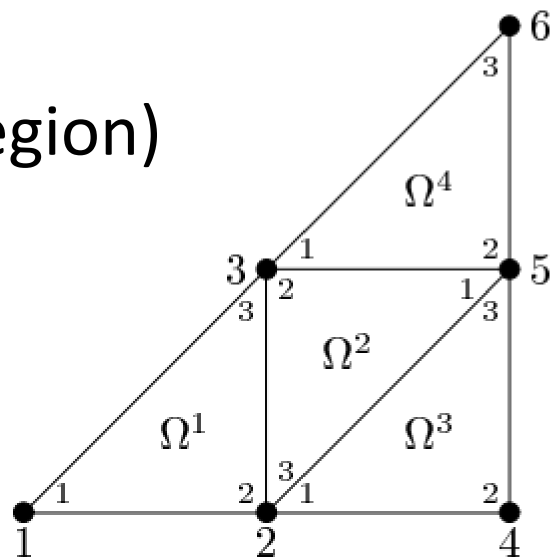
Finite Elements

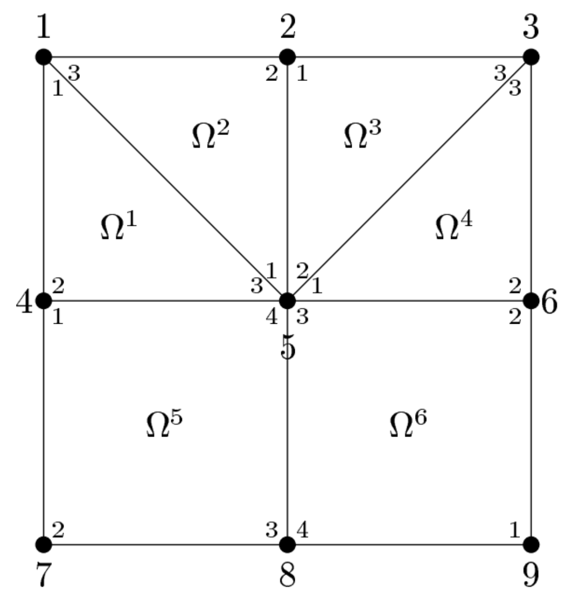
Element decomposition (meshing):

1-dim: (a line)



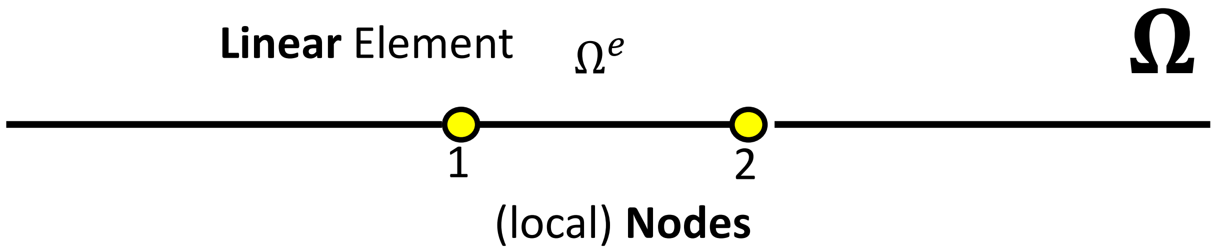
2-dim: (a 2D region)

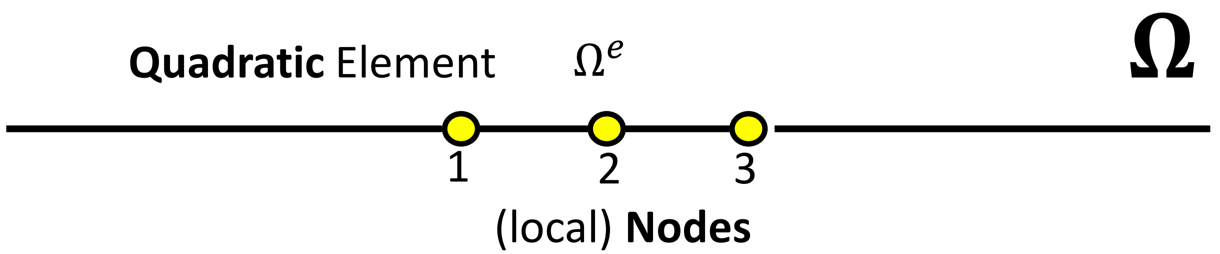




Finite Elements 1D

- For 1D domains, generically, the **elements** are defined as segments $\Omega^e = [x_i, x_{i+1}]$ that covers de complete domain $\boldsymbol{\Omega}$.



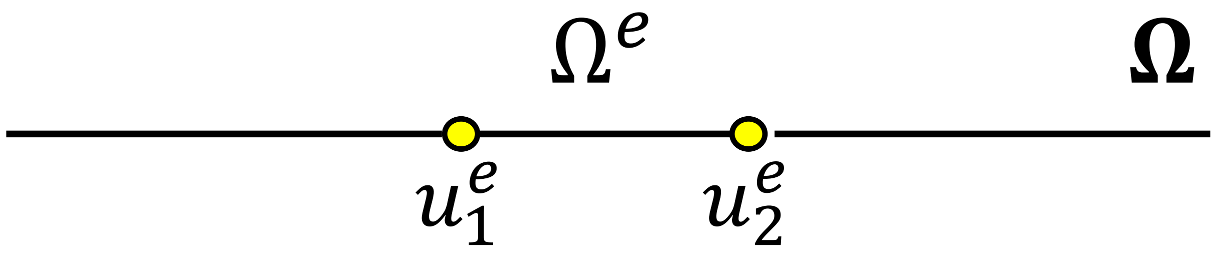


Finite Elements 1D

Let's assume that $u(x)$ is a **magnitude** (temperature, displacement, etc.) that we want to compute in the nodes n_i of one element Ω^e the usual FEM **notation** is:

$$u(n_i) = u_i^e$$

For the **linear** case:

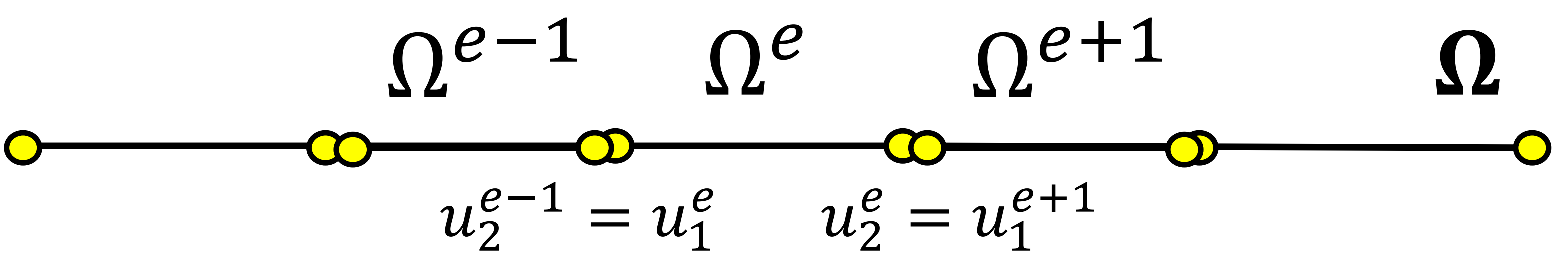


1D Elements Assembly

When we consider the total domain, nodes of consecutive (**connected**) elements must be identify in order to obtain *continuous solutions*:

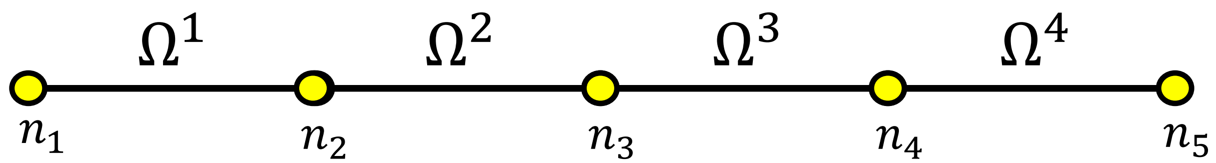
$$u_N^{e-1} = u_1^e, \qquad u_N^e = u_1^{e+1}$$

For the **linear** case (N =2):



1D Elements Assembly

- **Global enumeration**: Once we have identify the connected nodes, we rename them using a global enumeration.



Global Nodes

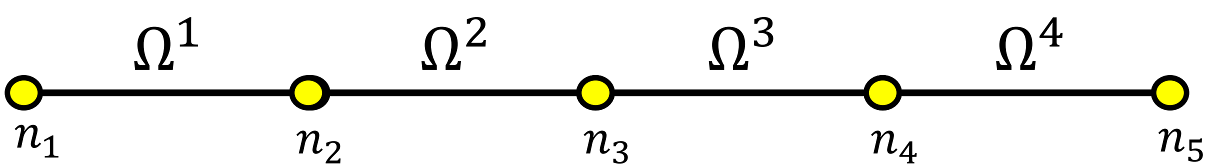
1D Elements Assembly

- **Connectivity Matrix**: Says the global nodes attached to each element.
- Example:

Node numbers

$$elem = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}$$

One row for each element



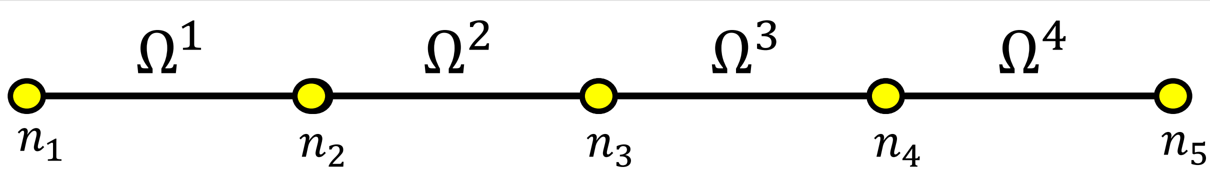
Global Nodes

1D Elements Assembly

- Example: Consider a bar of length $L = 8.$ Subdive it in 4 elements and give their coordinates.

Matlab

```
N= 4 ;  %number of divisions = number of elements
L = 8;  %total length
coordNodes = 0: L/N: L ; %compute the coordinates of the 5 nodes
numNod = size(coordNodes,2); numElem = N;
% alternatively:  coordNodes=linspace(0,L,N+1)
for i=1: numElem
      elem(i,:)=[i, i+1];
end
```



1D Elements Assembly

- **Stiffness Matrix** $(\boldsymbol{K})$: Is the matrix of the *linear system* that allows us to compute the magnitude values on each node.

Element Stiff Matrix $(\boldsymbol{K^e})$**:** Is the one related to the physical problem stated for each element (this is the *thought* part of the method).

Because it is associated to each element, its size agrees with the number of nodes in each element.

1-dim linear element (two nodes) ⟶ $\boldsymbol{K^e}$ is a 2x2 matrix

1-dim quadratic element (three nodes) ⟶ $\boldsymbol{K^e}$ is a 3x3 matrix

2-dim linear Triangular element (three nodes) ⟶ $\boldsymbol{K^e}$ is a 3x3 matrix

2-dim linear Quadrilateral element (four nodes) ⟶ $\boldsymbol{K^e}$ is a 4x4 mat

1D Elements Assembly

- **Notation**

1D **linear** elements:

$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{pmatrix}$$

1D **quadratic** elements

$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{pmatrix}$$

Usually K^e are **symmetric** matrices

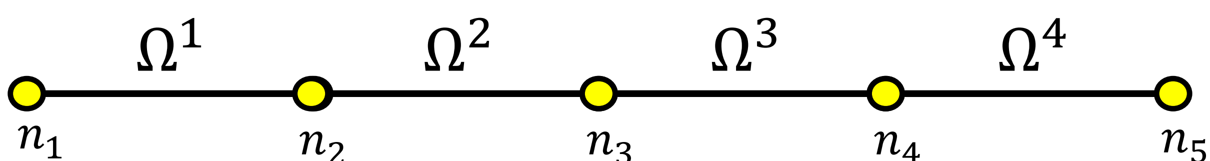
1D Elements Assembly

- **Global Stiff Matrix** $(\boldsymbol{K})$ **:** In a generic way, for a 1dim problem, the size of $\boldsymbol{K}$ is

numNod x **numNod**

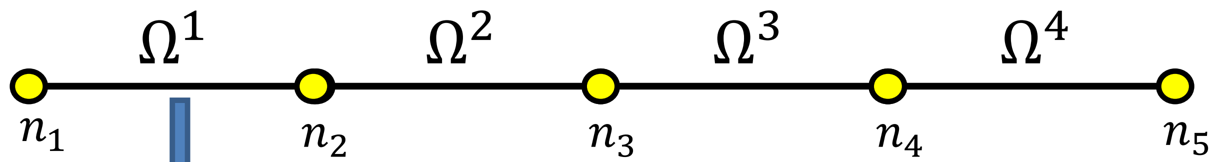
Example:

$$\boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$



1D Elements Assembly

- **Assembly**



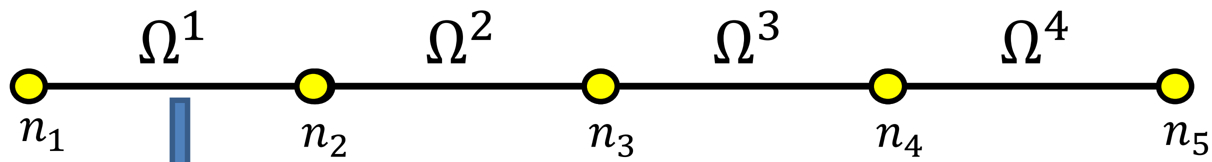
$$elem = \begin{pmatrix} 1 & 2 \\ \vdots & \vdots \\ 4 & 5 \end{pmatrix}$$

$$K^1 = \begin{pmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{pmatrix} \Rightarrow$$

$$\boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$

1D Elements Assembly

- **Assembly**



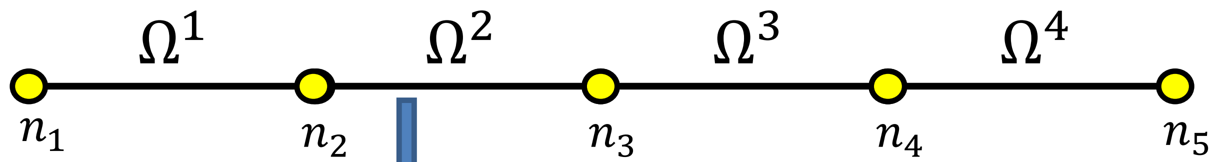
$$elem = \begin{pmatrix} 1 & 2 \\ \vdots & \vdots \\ 4 & 5 \end{pmatrix}$$

$$K^1 = \begin{pmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{pmatrix} \Rightarrow$$

$$\boldsymbol{K} = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13} & k_{14} & k_{15} \\ k_{21}^1 & k_{22}^1 & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$

1D Elements Assembly

- **Assembly**



$$elem = \begin{pmatrix} 1 & 2 \\ \vdots & \vdots \\ 4 & 5 \end{pmatrix}$$

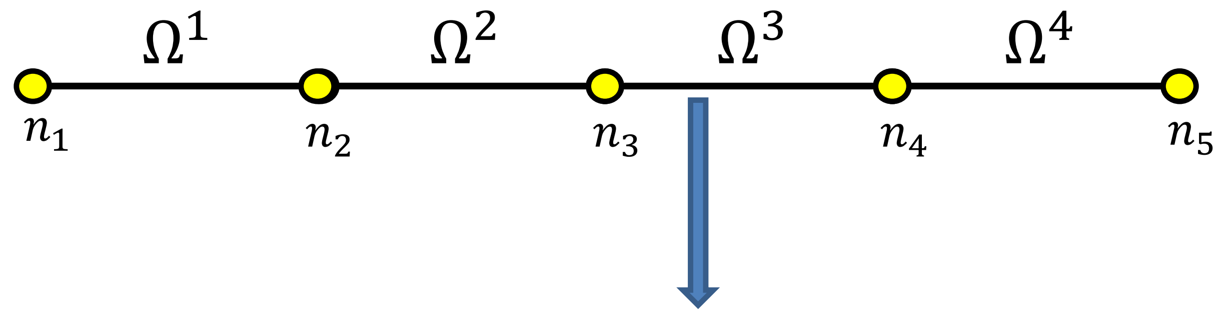
$$\boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$

Contribution of **two** elements

$$k_{22} = k_{22}^1 + k_{11}^2$$

1D Elements Assembly

- **Assembly**

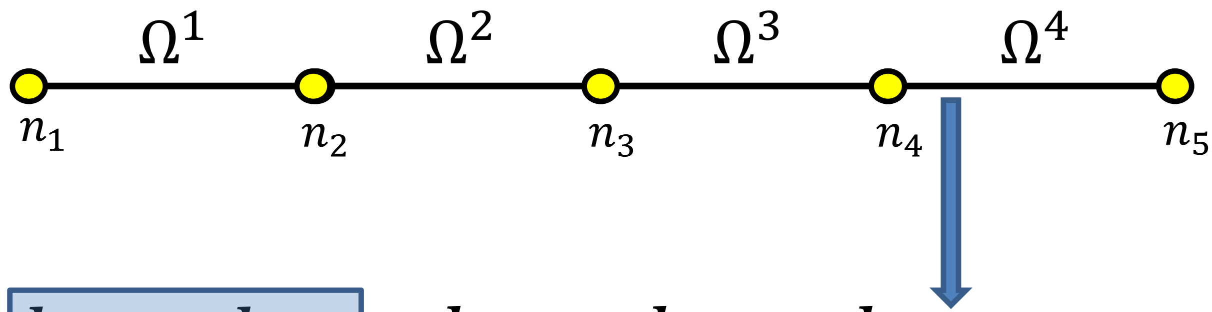


$$elem = \begin{pmatrix} 1 & 2 \\ \vdots & \vdots \\ 4 & 5 \end{pmatrix}$$

$$\boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$

1D Elements Assembly

- **Assembly**



$$elem = \begin{pmatrix} 1 & 2 \\ \vdots & \vdots \\ 4 & 5 \end{pmatrix}$$

$$\boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$

The rest of the elements in $\boldsymbol{K}$ are zero

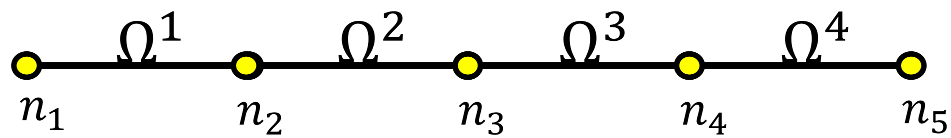
1D Elements Assembly

- **recovering** the previous example: Suppose that for each element its local Stiff matrix is constant (*we'll see later how to compute it*)

$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{pmatrix} = C \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Assembly 1D

```
Ke=C*[1,-1;-1,1]; %local stiff matrix
K=zeros(numNod); %initialize the global Stiff Matrix
for e=1: numElem
    rows=[elem(e,1); elem(e,2)];
    colums= rows;
    K(rows,colums)=K(rows,colums)+Ke; %assembly
end
```



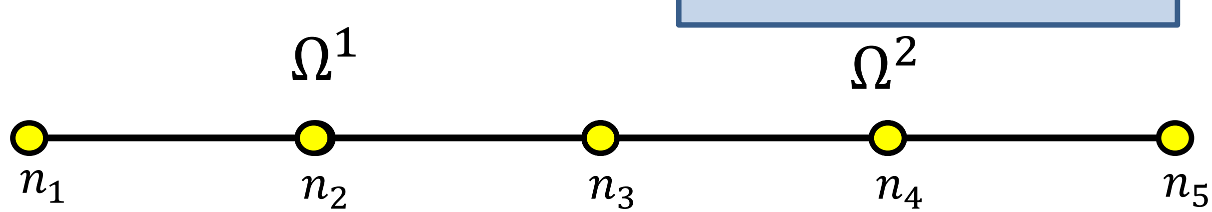
1D Elements Assembly

- **Exercise**: Consider the problem of the previous bar, but using now **quadratic elements.**

Modify the previous steps in order to obtain the assembly matrix when only two elements are taken.

$$K^e = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}$$

$$\boldsymbol{K} = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13} & k_{14} & k_{15} \\ k_{21}^1 & k_{22}^1 & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$

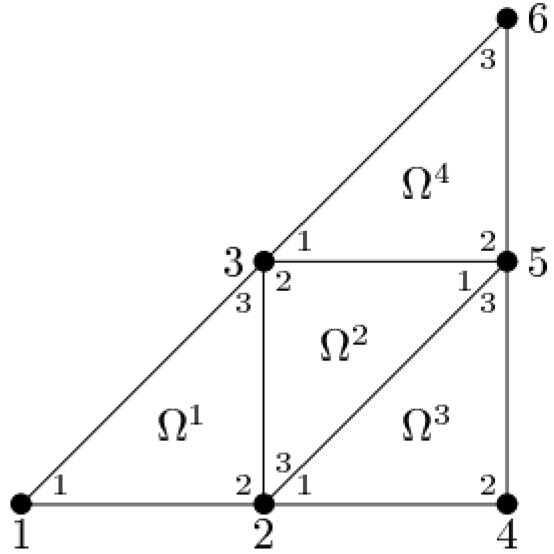


2D Elements Assembly

- Example of **local** and **global** nodes enumeration for linear triangular elements

Connectivity matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

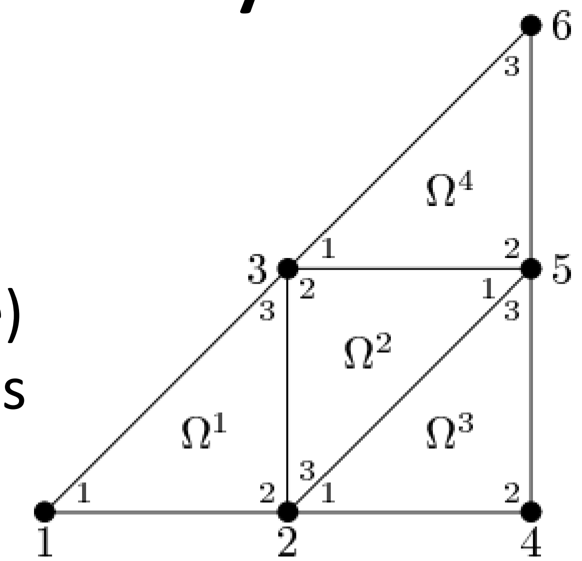


Hint: Notice that the local enumeration must be counter-clockwise in order to preserve orientation

2D Elements Assembly

- For linear triangular elements

If $\boldsymbol{u}(\boldsymbol{x})$ is a **1D magnitude** (temperature)
For each element the **Stiffness Matrix** is a 3x3 matrix



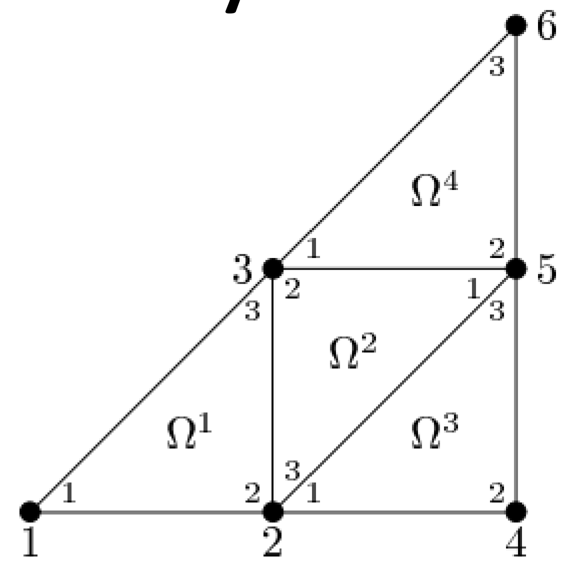
$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Exercise: do the assembly process for this example.

2D Elements Assembly

- For linear triangular elements

If $\boldsymbol{u}(\boldsymbol{x}) = (u_x, u_y)$ is a **2D magnitude** (displacements, fluid velocities, etc.) the **Stiffness Matrix** is 6x6 matrix.

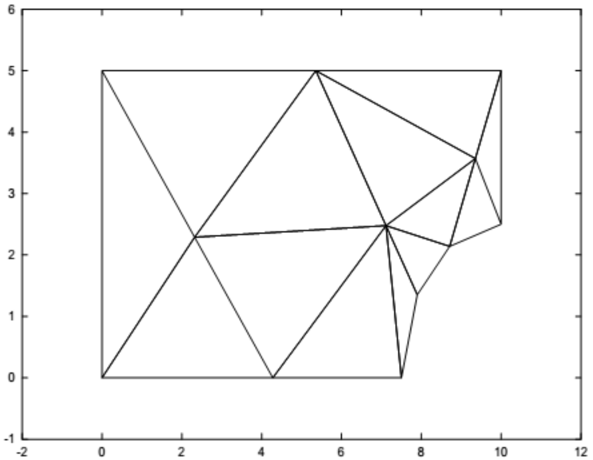


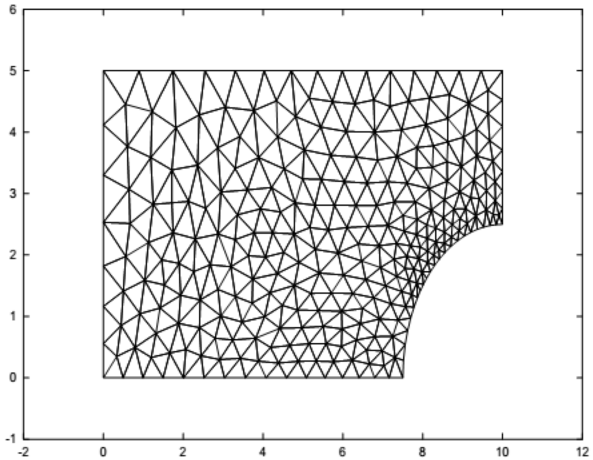
$$K^e = \begin{pmatrix} k_{11}^e & \cdots & k_{16}^e \\ \vdots & \ddots & \vdots \\ k_{61}^e & \cdots & k_{66}^e \end{pmatrix}, \qquad u = \begin{pmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{pmatrix}$$

Question: Which are the dimensions for quadrilaterals?

Meshing 2D domains

- Meshing a general domain is a **difficult problem**. We'll not study it in depth.
- Two main concerns when meshing a domain are:
  - Good fitting of the domain
  - Good Numerical properties (stability)

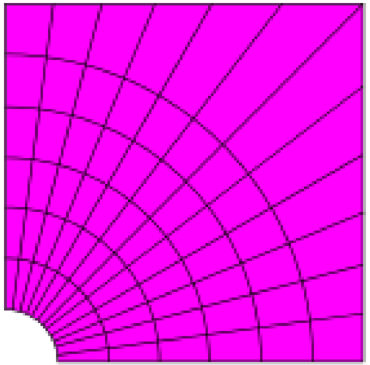




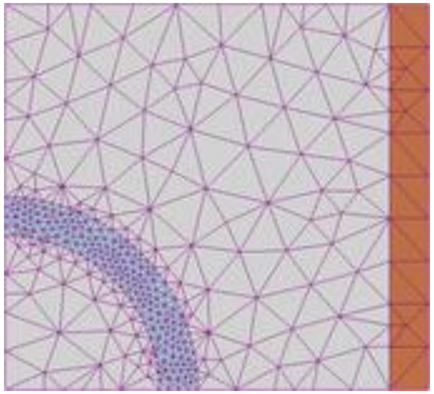
Meshing 2D domains

- Classification:
  - Structured Mesh:

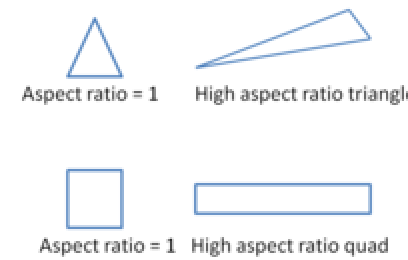
  are identified by regular connectivity



  - Unstructured Mesh



Meshing 2D domains

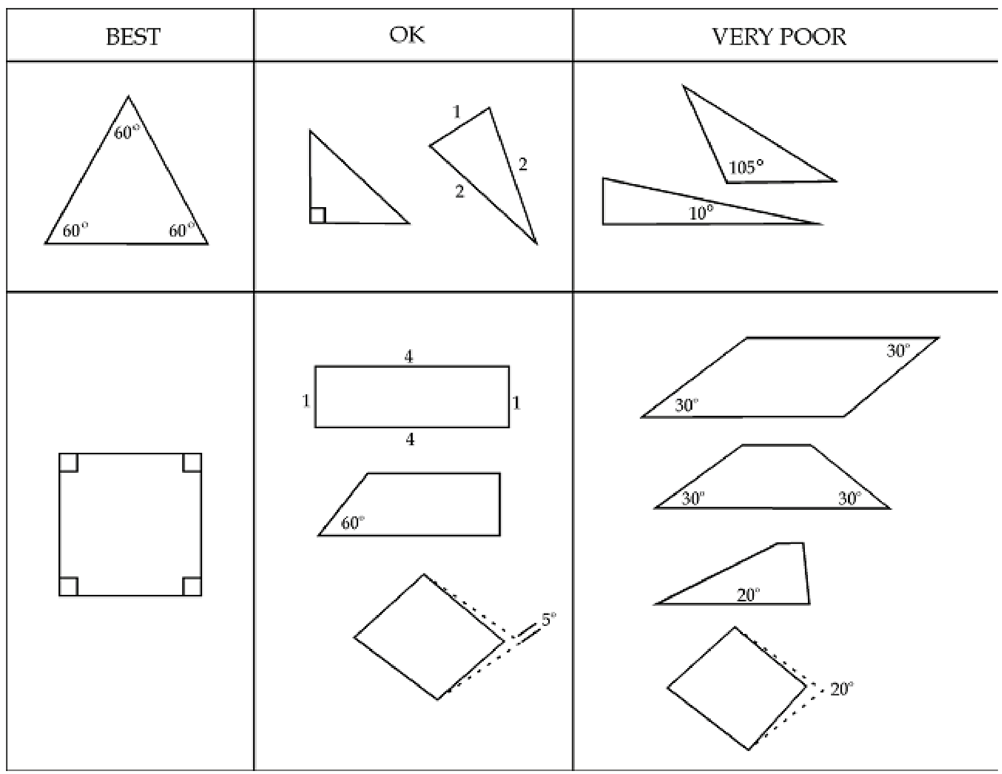


Mesh quality:

– **Aspect Ratio**: It is the ratio of *longest* to the *shortest* side in an element.

Best = 1

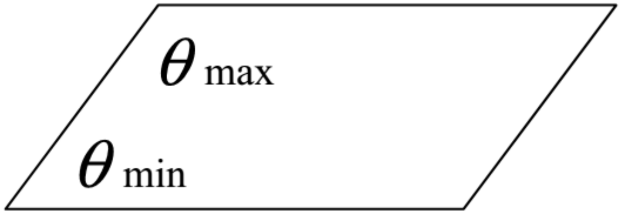
Acceptable < 5



Meshing 2D domains

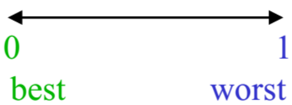
Mesh quality:

– Skewness:



Another common measure of quality is based on equiangular skew.

$$\text{Equiangle Skew} = \max\left[\frac{\theta_{max}-\theta_e}{180-\theta_e}, \frac{\theta_e-\theta_{min}}{\theta_e}\right]$$



where:

θ_{max} is the largest angle in a face or cell,

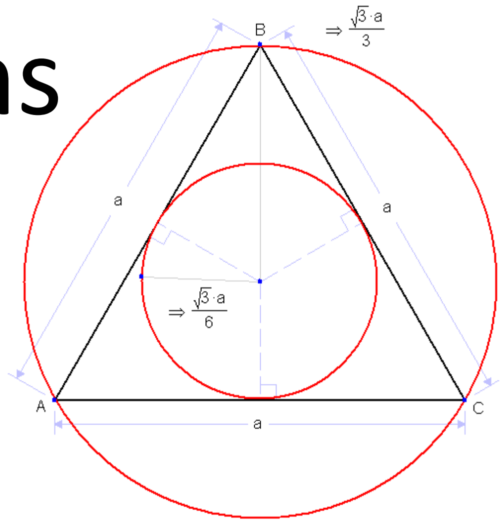
θ_{min} is the smallest angle in a face or cell,

θ_e is the angle for equi-angular face or cell i.e. 60 for a triangle and 90 for a square.

Value of Skewness	0-0.25	0.25-0.50	0.50-0.80	0.80-0.95	0.95-0.99	0.99-1.00
Cell Quality	excellent	good	acceptable	poor	sliver	degenerate

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria

Meshing 2D domains



Mesh quality:

- **Inscribed-Circumscribed ratio:**

$$q = 2\frac{r_{\text{in}}}{r_{\text{out}}} = \frac{(b+c-a)(c+a-b)(a+b-c)}{abc}$$

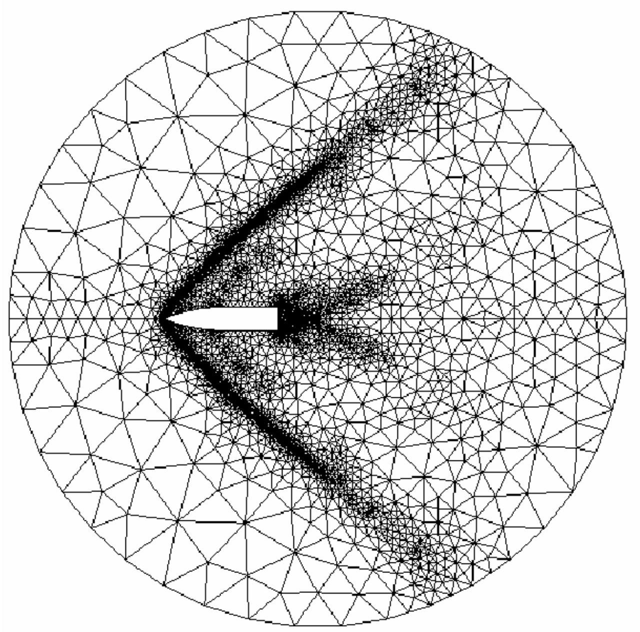
where a, b, c are the side lengths.

An equilateral triangle has $q = 1$

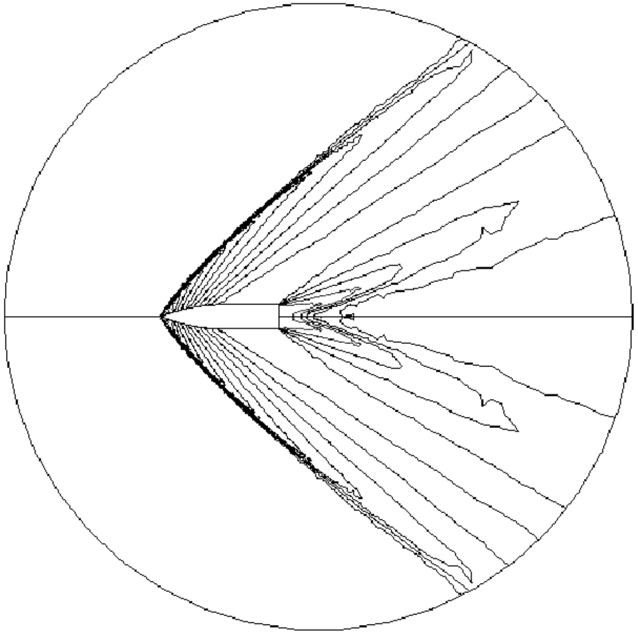
As a rule of thumb, if all triangles have $q > 0.5$ the results are good.

Meshing 2D domains

- **Mesh refinement**: More elements where physical features are changing



2D planar shell - final grid



2D planar shell - contours of pressure final grid

2D-Finite Elements

Finite Elements

Stp1: Discretize in **elements** ?

Stp2: Write the **variational** equations → **Weak form**

Stp3: Build the Linear System
Impose the BC

Stp4: Get **nodes** solution

Stp5: Extent solution to the Domain

2D-Model Equation

For 2D problems we will use the **model equation**. A 2on order PDE for $u = u(x,y)$ *(primary variable)*

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}u = f,$$

Defined on a 2-dim domain Ω, with $a_{ij}(x,y)$ and $f(x,y)$ known functions.

2D-Model Equation

- **Notation:** In many books you can find the expressions

$$\nabla \cdot u \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\,, \qquad \text{if} \quad u = u(x,y)$$

$$\nabla \cdot (u_1, u_2) \equiv div(u) \equiv \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\,, \qquad \text{if} \quad u = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix}$$

$$\nabla u \equiv grad(u) \equiv \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \qquad \text{if} \quad u = u(x,y)$$

- **Example: Poisson equation**

$$-\nabla \cdot (a\nabla u) = f$$

If a =const, $$-a\nabla \cdot (\nabla u) \equiv -a\,\nabla^2 u \equiv -a\Delta u = f$$

2D-Model Equation

- **Poisson equation:** It corresponds to the *model equation* with $-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+a_{00}u=f,$

$$a_{11}=a_{22}=a, \quad a_{12}=a_{21}=a_{00}=0$$

$$-\nabla\cdot(a\nabla u)=f$$

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a\frac{\partial u}{\partial y}\right)=f.$$

$$-a\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f$$

• Poisson Equation:

Some examples of the Poisson equation $-\nabla \cdot (k\nabla u) = f$
Natural boundary condition: $k\, \partial u/\partial n + \beta(u - u_\infty) = q$. Essential boundary condition: $u = \hat{u}$

Field of application	Primary variable u	Material constant k	Source variable f	Secondary variables $q, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$
1. Heat transfer	Temperature T	Conductivity k	Heat source Q	Heat flow q [comes from conduction $k\, \partial T/\partial n$ and convection $h(T - T_\infty)$]
2. Irrotational flow of an ideal fluid	Stream function ψ	Density ρ	Mass production σ (normally zero)	Velocities: $\frac{\partial \psi}{\partial x} = -v$, $\frac{\partial \psi}{\partial y} = u$
	Velocity potential ϕ	Density ρ	Mass production σ (normall zero)	$\frac{\partial \phi}{\partial x} = u$, $\frac{\partial \phi}{\partial y} = v$
3. Groundwater flow	Piezometric head ϕ	Permeability K	Recharge Q (or pumping, $-Q$)	Seepage $q = k\frac{\partial \phi}{\partial n}$ Velocities: $u = -k\frac{\partial \phi}{\partial x}$, $v = -k\frac{\partial \phi}{\partial y}$
4. Torsion of members with constant cross-section	Stress function Ψ	$k = 1$	$f = 2$	$G\theta\frac{\partial \Psi}{\partial x} = -\sigma_{yz}$
		G = shear modulus	θ = angle of twist per unit length	$G\theta\frac{\partial \Psi}{\partial y} = \sigma_{xz}$
5. Electrostatics	Scalar potential ϕ	Dielectric constant ε	Charge density ρ	Displacement flux density D_n
6. Magnetostatics	Magnetic potential ϕ	Permeability μ	Charge density ρ	Magnetic flux density B_n
7. Transverse deflection of elastic membranes	Transverse deflection u	Tension T in membrane	Transversely distributed load	Normal force q

Table from Reddy's *FEM book McGraw-Hill*

Weak Formulation

- Using a general weight function $\omega = \omega(x, y)$, we write the integral expression:

$$\int_{\Omega^k} \omega(x,y) \left[-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} \right) + a_{00}u - f \right] dx\, dy = 0\,.$$

In 2D, we have the **divergence theorem** (from Gauss)

$$\int_{\Omega^k} \operatorname{div} \vec{G}\, dx\, dy = \int_{\partial\Omega^k} \vec{G} \cdot \vec{n}\, d\ell,$$

$\partial\Omega^k$ is the boundary of the domain Ω^k, and $\vec{n}$ is the normal vector to $\partial\Omega$ (pointing external)

Weak Formulation

- We need to introduce some notation, let's say

$$F_1 = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y},$$

and

$$F_2 = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}$$

Derivating now the products ωF_1 and ωF_2 we get:

$$-\omega\frac{\partial F_1}{\partial x} = \frac{\partial \omega}{\partial x}F_1 - \frac{\partial}{\partial x}(\omega F_1)\,, \qquad -\omega\frac{\partial F_2}{\partial y} = \frac{\partial \omega}{\partial y}F_2 - \frac{\partial}{\partial y}(\omega F_2)\,,$$

Then, the first part of the weak form says

$$\int_{\Omega^k}\omega\left(-\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}\right)dx\,dy = \int_{\Omega^k}\left(\frac{\partial \omega}{\partial x}F_1 + \frac{\partial \omega}{\partial y}F_2 - \frac{\partial}{\partial x}(\omega F_1) - \frac{\partial}{\partial y}(\omega F_2)\right)dx\,dy\,.$$

Weak Formulation

- The last two terms are the divergence terms. Therefore,

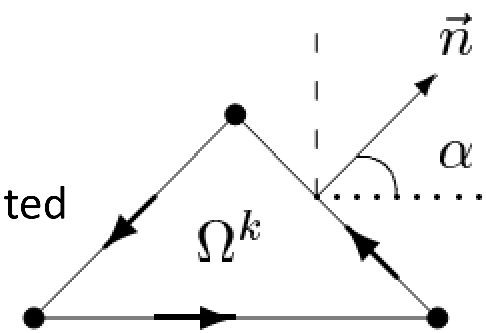
$$\int_{\Omega^k}\left[\frac{\partial \omega}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial \omega}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)\right]dx\,dy\,-$$

$$-\int_{\Gamma^k}\omega\left[n_x\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+n_y\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)\right]d\ell,$$

where $\Gamma^k = \partial\Omega^k$..

For a triangle is:

Γ^k positive oriented



$$\vec{n} = (n_x\,, n_y) = (\cos\alpha\,, \sin\alpha)$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria Industrial de Barcelona
ETSEIB

Weak Formulation

- The terms associated to the secondary variables are:

$$q_n \equiv n_x\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+n_y\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{pmatrix}\cdot\vec{n},$$

- Finally using this definition we rewrite the weak form:

$$\int_{\Omega^k}\left[\frac{\partial \omega}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial \omega}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+a_{00}u\omega-\omega f\right]dxdy-\int_{\Gamma^k}\omega q_n d\ell=0\,.$$

- Now using the shape functions $\omega=\psi_i^k(x,y),\ i=1\ldots n.$

and $u(x,y)=\sum_{j=1}^{n}u_j^k\psi_j^k(x,y),$ we get:

$$\int_{\Omega^k}\left[\frac{\partial \psi_i^k}{\partial x}\left(a_{11}\sum_{j=1}^{n}u_j^k\frac{\partial \psi_j^k}{\partial x}+a_{12}\sum_{j=1}^{n}u_j^k\frac{\partial \psi_j^k}{\partial y}\right)+\frac{\partial \psi_i^k}{\partial y}\left(a_{21}\sum_{j=1}^{n}u_j^k\frac{\partial \psi_j^k}{\partial x}+a_{22}\sum_{j=1}^{n}u_j^k\frac{\partial \psi_j^k}{\partial y}\right)+\right.$$

$$\left.+a_{00}\psi_i^k\sum_{j=1}^{n}u_j^k\psi_j^k-\psi_i^k f\right]dx\,dy-\int_{\Gamma^k}\psi_i^k q_n d\ell=0,\quad i=1\ldots n.$$

Weak Formulation

- Grouping the unknown terms u_j^k

$$\sum_{j=1}^{n}\left[\int_{\Omega^k}\left[\frac{\partial\psi_i^k}{\partial x}\left(a_{11}\frac{\partial\psi_j^k}{\partial x}+a_{12}\frac{\partial\psi_j^k}{\partial y}\right)+\frac{\partial\psi_i^k}{\partial y}\left(a_{21}\frac{\partial\psi_j^k}{\partial x}+a_{22}\frac{\partial\psi_j^k}{\partial y}\right)+a_{00}\psi_i^k\psi_j^k\right]dxdy\right]u_j^k-$$

$$-\int_{\Omega^k} f\psi_i^k\,dx\,dy-\int_{\Gamma^k}\psi_i^k q_n\,d\ell=0,\quad i=1\ldots n.$$

or, as a **linear system** $\sum_{j=1}^{n} K_{ij}^k u_j^k = F_i^k + Q_i^k,\ i=1\ldots n.$

$$K_{ij}^k=\int_{\Omega^k}\left[\frac{\partial\psi_i^k}{\partial x}\left(a_{11}\frac{\partial\psi_j^k}{\partial x}+a_{12}\frac{\partial\psi_j^k}{\partial y}\right)+\frac{\partial\psi_i^k}{\partial y}\left(a_{21}\frac{\partial\psi_j^k}{\partial x}+a_{22}\frac{\partial\psi_j^k}{\partial y}\right)+a_{00}\psi_i^k\psi_j^k\right]dx\,dy,$$

$$F_i^k=\int_{\Omega^k} f\psi_i^k\,dx\,dy,\qquad Q_i^k=\int_{\Gamma^k} q_n\psi_i^k\,d\ell.$$

Weak Formulation

$$q_n \equiv n_x\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+n_y\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)$$

Notation:

$u = u(x, y)$ is named **primary variable**

q_n is named **secondary variable**

Boundary Conditions (BC):

$u_A = u(x_A)$ is an **essential BC** (fix the primary variable)

$q_n = Q_0$ is a **natural BC** (fix the secondary variable)

Weak Formulation

Notation from the global system of equations:

$$[K^k]u^k = F^k + Q^k.$$

Here we will use

$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

with

$$K_{ij}^{k,00} = \int_{\Omega^k} a_{00}\, \psi_i^k \psi_j^k \, dx\, dy\,, \qquad K_{ij}^{k,11} = \int_{\Omega^k} a_{11} \frac{\partial \psi_i^k}{\partial x} \frac{\partial \psi_j^k}{\partial x} dx\, dy\,,$$

$$K_{ij}^{k,12} = \int_{\Omega^k} a_{12} \frac{\partial \psi_i^k}{\partial x} \frac{\partial \psi_j^k}{\partial y} dx\, dy\,, \qquad K_{ij}^{k,21} = \int_{\Omega^k} a_{21} \frac{\partial \psi_i^k}{\partial y} \frac{\partial \psi_j^k}{\partial x} dx\, dy\,,$$

$$K_{ij}^{k,22} = \int_{\Omega^k} a_{22} \frac{\partial \psi_i^k}{\partial y} \frac{\partial \psi_j^k}{\partial y} dx\, dy\,.$$

Computing the Integrals

To compute terms like these ones:

$$K_{ij}^{k,00} = \int_{\Omega^k} a_{00}\, \psi_i^k \psi_j^k dx\, dy\,, \qquad K_{ij}^{k,11} = \int_{\Omega^k} a_{11}\, \frac{\partial \psi_i^k}{\partial x} \frac{\partial \psi_j^k}{\partial x} dx\, dy\,,$$

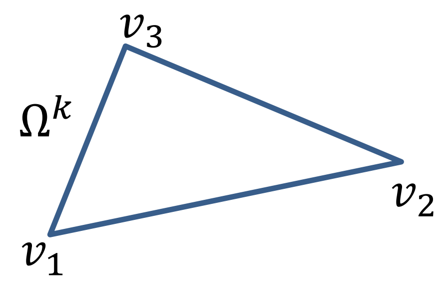
we need to compute **numerically** these 2D integrals. For that we will use **Gauss integration methods** that will be introduced later.

For some easy cases there are some **explicit formulas** that we present next.

Computing the Integrals: Triangles

- If we consider **constant coefficients** for the *model equation*

In the case of a general **linear triangular element**



$$\boldsymbol{K}_{ij}^{k,11} = a_{11} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial x}(x,y) dxdy = a_{11} \frac{1}{4A_k} \beta_i \beta_j$$

$$\boldsymbol{K}_{ij}^{k,12} = a_{12} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial y}(x,y) dxdy = a_{12} \frac{1}{4A_k} \beta_i \gamma_j$$

$$\boldsymbol{K}_{ij}^{k,21} = a_{21} \frac{1}{4A_k} \gamma_i \beta_j$$

$$\boldsymbol{K}_{ij}^{k,22} = a_{22} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial y}(x,y) \frac{\partial \psi_j^k}{\partial y}(x,y) dxdy = a_{22} \frac{1}{4A_k} \gamma_i \gamma_j$$

$$\boldsymbol{K}_{ij}^{k,00} = a_{00} \iint_{\Omega_k} \psi_i^k(x,y) \psi_j^k(x,y) dxdy = a_{00} \frac{A_k}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$F^k = \frac{f_k A_k}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

If the vertices of the triangle are $v_i = (x_i, y_i)$ we define:

$$\beta_i = y_j - y_k$$

$$\gamma_i = -(x_j - x_k)$$

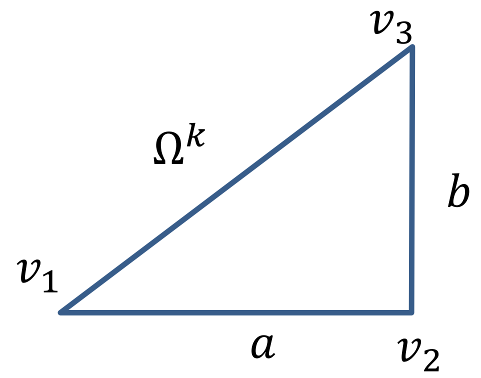
(i, j, k) cyclic permutations

A_k is triangle area

Computing the Integrals: Triangles

- In the case of a general **linear triangular rectangle element** for the **Poisson's Equation**

$$(a_{11} = a_{22} = c, \quad a_{12} = a_{21} = a_{00} = 0)$$



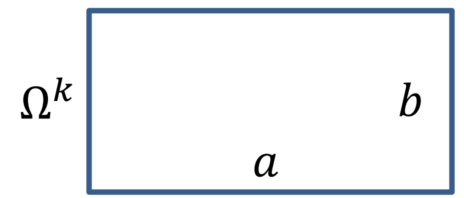
$$K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2 + b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}$$

$$F^k = \frac{f_k A_k}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{f_k ab}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Computing the Integrals: Rectangles

- If we consider **constant coefficients** for the *model equation*

In the case of a **rectangular quadrilateral**



$$[K^{k,11}] = \frac{b\,a^k_{11}}{6a}\begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix}, \quad [K^{k,12}] = \frac{a^k_{12}}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$[K^{k,22}] = \frac{a\,a^k_{22}}{6b}\begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix}, \quad [K^{k,00}] = \frac{ab\,a^k_{00}}{36}\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}.$$

The 1D case: Variational Formulation

- Similar to the 2D case, the **model equation** for the 1D case is: $$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$
- If $\Omega^k = [x_A, x_B]$, the variational formulation gives:

$$\sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx\right]u_j^k + \sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_0\psi_i^k\psi_j^k\,dx\right]u_j^k - \int_{x_A}^{x_B} f\psi_i^k\,dx - Q_i^k = 0,$$

Like in 2D, the term $Q \equiv a_1\frac{du}{dx}$, and because the outer normal orientation, we have $Q_i^k = -a_1\frac{du}{dx}|_{x=X_A}$ and $Q_i^k = a_1\frac{du}{dx}|_{x=X_B}$ Notice the minus sign on the left node.

The 1D case: Variational Formulation

- In compact form:

$$\sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx\right] u_j^k + \sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_0 \psi_i^k \psi_j^k\,dx\right] u_j^k - \int_{x_A}^{x_B} f\psi_i^k\,dx - Q_i^k = 0,$$

$$\sum_{j=1}^{n} K_{ij}^k u_j^k - F_i^k - Q_i^k = 0\,, \quad i = 1\ldots n$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria Industrial de Barcelona
ETSEIB

The 1D case: Variational Formulation

- In compact form:

$$\sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx\right]u_j^k + \sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_0 \psi_i^k \psi_j^k\,dx\right]u_j^k - \int_{x_A}^{x_B} f\psi_i^k\,dx - Q_i^k = 0,$$

$$\sum_{j=1}^{n} K_{ij}^k u_j^k - F_i^k - Q_i^k = 0\,,\quad i = 1\ldots n$$

$$K_{ij}^k = K_{ij}^{k,1} + K_{ij}^{k,0}$$

Notice that now these are 1D integrals

$$K_{ij}^{k,1} = \int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx, \qquad K_{ij}^{k,0} = \int_{x_A}^{x_B} a_0 \psi_i^k \psi_j^k\,dx, \qquad F_i^k = \int_{x_A}^{x_B} f\psi_i^k\,dx.$$

The 1D Linear element

- Consider now the **linear reference element** $\Omega^R = [-1,1]$.
- The shape functions can be written for $\xi \in [-1,1]$

$$\psi_1^R(\xi) = \frac{1}{2}(1-\xi), \qquad \psi_2^R(\xi) = \frac{1}{2}(1+\xi).$$

- The idea is to use **integral properties** to pass every other linear element to the reference one
- For a general element $\Omega^k = [x_A, x_B]$, with $x \in [x_A, x_B]$

$$x = \phi_k(\xi) \longrightarrow \phi_k(\xi) = \frac{h_k}{2}(\xi+1) + x_A, \qquad h_k = x_B - x_A$$

$$\xi = \phi_k^{-1}(x) \longrightarrow \phi_k^{-1}(x) = \frac{2}{h_k}(x - x_A) - 1.$$

The 1D Linear element

- Therefore:

$$K_{ij}^{k,1} = \int_{x_A}^{x_B} a_1(x)\frac{d\psi_i^k(x)}{dx}\frac{d\psi_j^k(x)}{dx}\,dx = \int_{-1}^{1} a_1(\phi_k(\xi))\,\frac{d\psi_i^R(\xi)}{d\xi}\frac{2}{h_k}\,\frac{d\psi_j^R(\xi)}{d\xi}\frac{2}{h_k}\,\frac{h_k}{2}\,d\xi,$$

$$K_{ij}^{k,0} = \int_{x_A}^{x_B} a_0(x)\,\psi_i^k(x)\psi_j^k(x)\,dx = \int_{-1}^{1} a_0(\phi_k(\xi))\,\psi_i^R(\xi)\psi_j^R(\xi)\frac{h_k}{2}d\xi.$$

with

$$\psi_1^R(\xi) = \frac{1}{2}(1-\xi), \qquad \psi_2^R(\xi) = \frac{1}{2}(1+\xi).$$

The 1D Linear element

- The **constant case**:

Consider now the case where $a_1(x) = a_1^k, a_o(x) = a_0^k,\ f(x) = f^k$

$$K_{11}^{k,1} = \int_{-1}^{1} \frac{a_1^k}{h_k^2} \frac{h_k}{2}\, d\xi = \frac{a_1^k}{h_k}, \qquad K_{11}^{k,0} = \int_{-1}^{1} a_0^k \frac{(1-\xi)^2}{4} \frac{h_k}{2}\, d\xi = \frac{a_0^k h_k}{3},$$

$$K_{12}^{k,1} = K_{21}^{k,1} = \int_{-1}^{1} \frac{-a_1^k}{h_k^2} \frac{h_k}{2}\, d\xi = \frac{-a_1^k}{h_k}, \qquad K_{12}^{k,0} = K_{21}^{k,0} = \int_{-1}^{1} a_0^k \frac{(1-\xi)(1+\xi)}{4} \frac{h_k}{2}\, d\xi = \frac{a_0^k h_k}{6},$$

$$K_{22}^{k,1} = \int_{-1}^{1} \frac{a_1^k}{h_k^2} \frac{h_k}{2}\, d\xi = \frac{a_1^k}{h_k}, \qquad K_{22}^{k,0} = \int_{-1}^{1} a_0^k \frac{(1+\xi)^2}{4} \frac{h_k}{2}\, d\xi = \frac{a_0^k h_k}{3}.$$

$$F_1^k = \int_{-1}^{1} \left(f^k \frac{1-\xi}{2} \right) \frac{h_k}{2} d\xi = \frac{1}{2} f^k h_k, \qquad F_2^k = \int_{-1}^{1} \left(f^k \frac{1+\xi}{2} \right) \frac{h_k}{2} d\xi = \frac{1}{2} f^k h_k.$$

The 1D Linear element

- The **constant case**:

collecting all the terms we have

$$[K^{k,1}] = \frac{a_1^k}{h_k}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$[K^{k,0}] = \frac{a_0^k h_k}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$F^k = \frac{f^k h_k}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The 1D Quadratic element

- When we consider quadratic elements, the shape functions are

$$\psi_1^R(\xi) = \frac{1}{2}\xi(\xi - 1), \qquad \psi_2^R(\xi) = (1+\xi)(1-\xi), \qquad \psi_3^R(\xi) = \frac{1}{2}\xi(1+\xi)$$

- In the **constant coefficient** case we obtain:

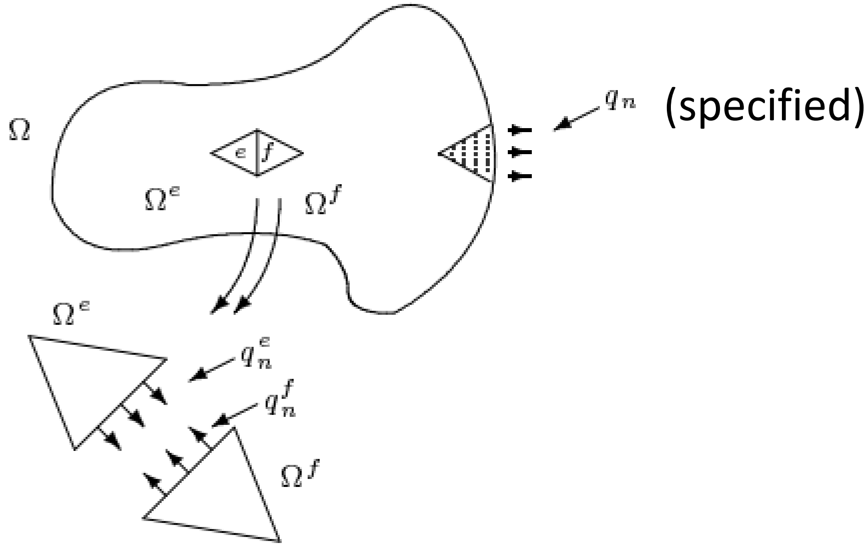
$$[K^{k,1}] = \frac{a_1^k}{3h_k}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}$$

$$F^k = \frac{f^k h_k}{6}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

$$[K^{k,0}] = \frac{a_0^k h_k}{30}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}$$

Boundary Conditions

- In 2D the boundary conditions (BC) are slightly different. Balance, of course, applies to interior faces and only the ones on the boundary have to be consider



Boundary Conditions

- The integrals we have to compute are of the form

$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell.$$

where Γ_k is the boundary of the element Ω^k . If we consider a triangular element

$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = \int_{\Gamma_1^k} q_{n1}^k(s) \psi_{i1}^k(s) \, ds + \int_{\Gamma_2^k} q_{n2}^k(s) \psi_{i2}^k(s) \, ds + \int_{\Gamma_3^k} q_{n3}^k(s) \psi_{i3}^k(s) \, ds,$$

or $\quad Q_i^k \equiv Q_{i1}^k + Q_{i2}^k + Q_{i3}^k,$ with $\; Q_{ij}^k = \int_{\Gamma_j^k} q_{nj}^k(s) \psi_{ij}^k(s) \, ds,$

(Q_{ij}^k means the flux on node i corresponding to the contribution of edge j)

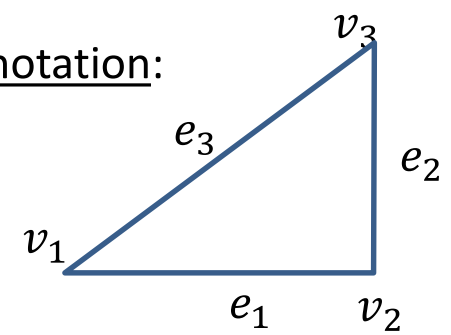
Boundary Conditions

- Here $q^k_{nj}(s)$ and $\psi^k_{ij}(s)$ are the restrictions of the general functions to the corresponding **edge** of the triangle.
- For the shape functions, they can be seen as the 1D **Lagrange's polynomial associated to the edge**

$$\begin{aligned}
&\psi^k_{11}(s) = 1 - \frac{s}{h^k_1}, && \psi^k_{12}(s) = 0, && \psi^k_{13}(s) = \frac{s}{h^k_3},\\
&\psi^k_{21}(s) = \frac{s}{h^k_1}, && \psi^k_{22}(s) = 1 - \frac{s}{h^k_2}, && \psi^k_{23}(s) = 0,\\
&\psi^k_{31}(s) = 0, && \psi^k_{32}(s) = \frac{s}{h^k_2}, && \psi^k_{33}(s) = 1 - \frac{s}{h^k_3}.
\end{aligned}$$

h^k_j is the length of the j-th edge of the triangle

Edge notation:



Boundary Conditions

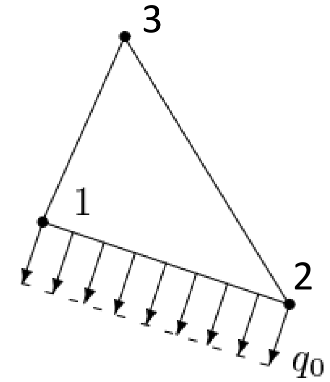
There are different types of BC, like the following cases:

- In case (a) the BC is only applied to an edge and it is **constant** q_0. The integral is restricted to

$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = q_0 \int_0^{h_1^k} \psi_{i1}^k(s)\, ds. \quad (i = 1,2,3)\,.$$

with

$$\psi_{11}^k(s) = (1 - \tfrac{s}{h_1^k}),\ \psi_{21}^k(s) = \tfrac{s}{h_1^k} \text{ i},\ \psi_{31}^k(s) = 0.$$



(a)

Computing the integrals we obtain

$$Q_1^k = Q_{11}^k,\ Q_2^k = Q_{21}^k \text{ i},\ Q_3^k = 0.$$

where

$$Q_{11}^k = \tfrac{1}{2} q_0 h_1^k,\ Q_{21}^k = \tfrac{1}{2} q_0 h_1^k,$$

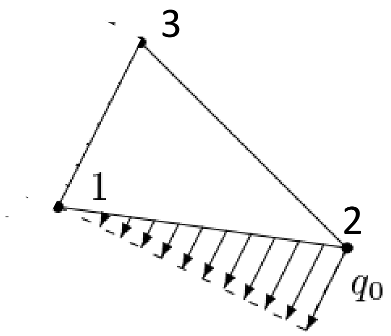
The **constant value is distributed** between nodes 1 and 2

Boundary Conditions

- Consider now a **linear function** applied

From node 2 to node 1. Now $q_{n1}^k(s) = q_0 s/h_1^k$

and



$$Q_{11}^k = \int_0^{h_1^k} q_0 \frac{s}{h_1^k}\left(1 - \frac{s}{h_1^k}\right) ds = \frac{1}{6} h_1^k q_0, \quad \text{i}, \quad Q_{21}^k = \int_0^{h_1^k} q_0 \frac{s}{h_1^k}\left(\frac{s}{h_1^k}\right) ds = \frac{1}{3} h_1^k q_0.$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria Industrial de Barcelona
ETSEIB

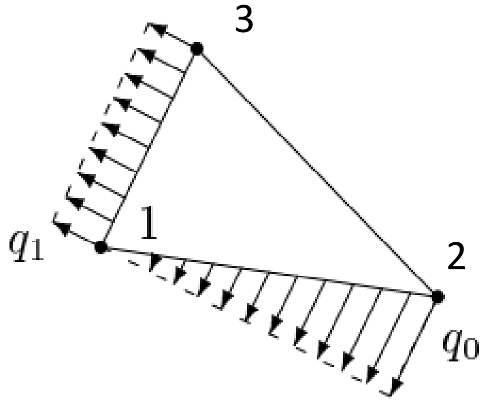
Boundary Conditions

- As a final case now we have de contribution of the two previous cases. In principle,

$$Q_1^k = Q_{11}^k + Q_{13}^k,$$
$$Q_2^k = Q_{21}^k + Q_{22}^k$$
$$Q_3^k = Q_{32}^k + Q_{33}^k.$$

with

$$Q_{23}^k = Q_{31}^k = 0.$$
$$Q_{13}^k = Q_{33}^k = q_1 h_3^k/2.$$
$$Q_{11}^k = \frac{1}{6} h_1^k q_0,$$
$$Q_{21}^k = \frac{1}{3} h_1^k q_0.$$



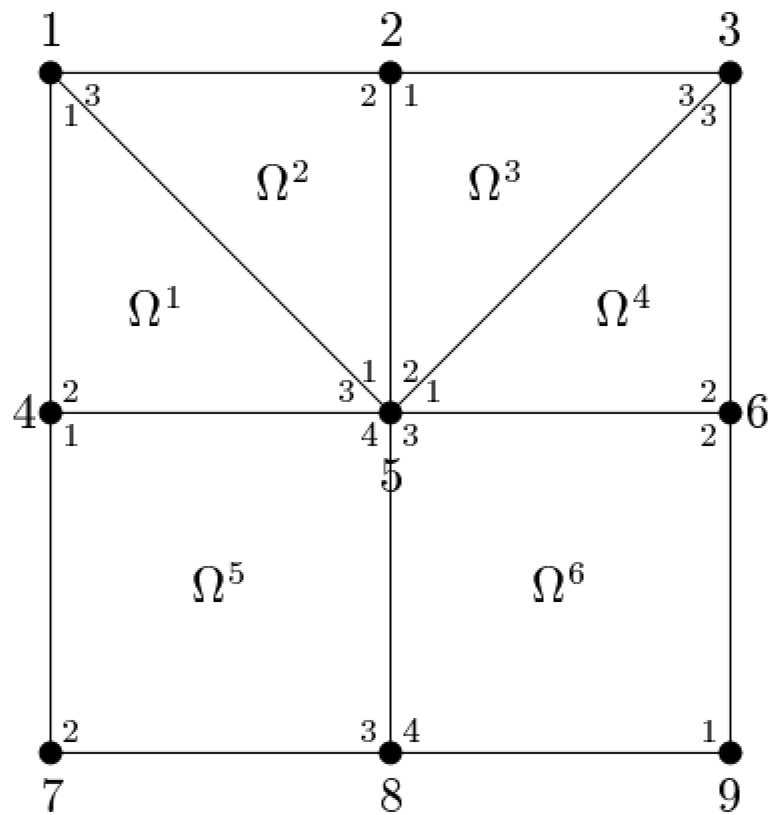
Finally
$$Q_1^k = \frac{1}{6} h_1^k q_0 + \frac{q_1 h_3^k}{2}$$
$$Q_2^k = \frac{1}{3} h_1^k q_0$$
$$Q_3^k = \frac{q_1 h_3^k}{2}$$

2D Assembly of element equations

- The assembly rules are very similar to the 1D case. Let's consider the example:

Although it is **not usual**, we can mix different type of elements:

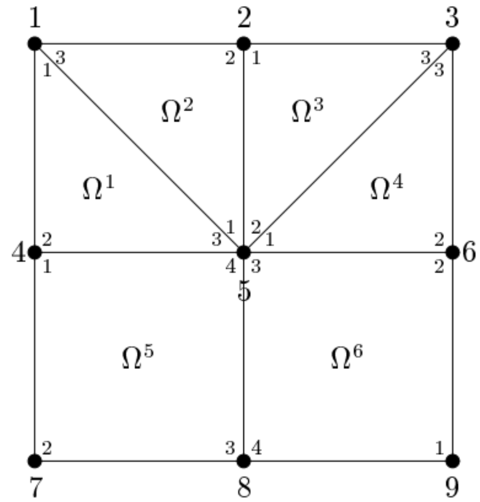
Triangular + Rectangular



2D Assembly of element equations

- The connectivity matrix in this case is not uniform

$$C = \begin{pmatrix} 1 & 4 & 5 & * \\ 5 & 2 & 1 & * \\ 2 & 5 & 3 & * \\ 5 & 6 & 3 & * \\ 4 & 7 & 8 & 5 \\ 9 & 6 & 5 & 8 \end{pmatrix}$$



Triangular

$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{pmatrix}, \quad e=1,2,3,4$$

Rectangular

$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e & k_{13}^e & k_{14}^e \\ k_{21}^e & k_{22}^e & k_{23}^e & k_{24}^e \\ k_{31}^e & k_{32}^e & k_{33}^e & k_{34}^e \\ k_{41}^e & k_{42}^e & k_{43}^e & k_{44}^e \end{pmatrix}, \quad e=5,6$$

2D Assembly of element equations

- The global **stiffness matrix** is

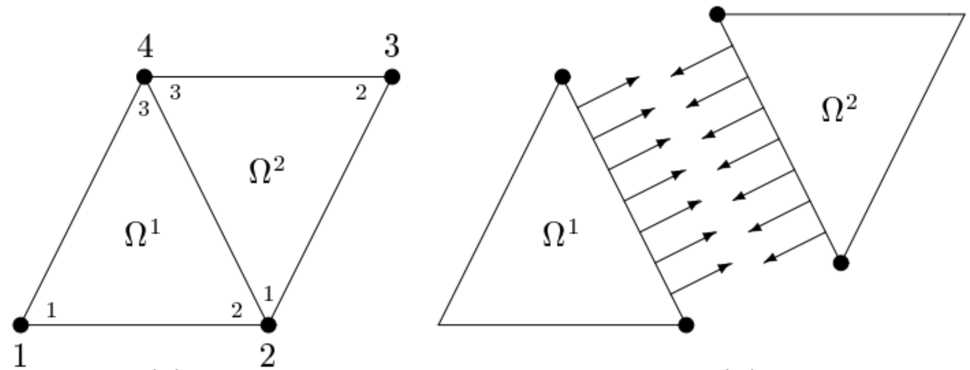
$$
\begin{pmatrix}
K^1_{11}+K^2_{33} & K^2_{11} & 0 & K^1_{12} & K^1_{13}+K^2_{31} & 0 & 0 & 0 & 0 \\
K^2_{13} & K^2_{22}+K^3_{11} & K^3_{13} & 0 & K^2_{21}+K^3_{12} & 0 & 0 & 0 & 0 \\
0 & K^3_{31} & K^3_{33}+K^4_{33} & 0 & K^3_{32}+K^4_{31} & K^4_{32} & 0 & 0 & 0 \\
K^1_{21} & 0 & 0 & K^1_{22}+K^5_{11} & K^1_{23}+K^5_{14} & 0 & K^5_{12} & K^5_{13} & 0 \\
K^1_{31}+K^2_{13} & K^2_{12}+K^3_{21} & K^3_{23}+K^4_{13} & K^1_{32}+K^5_{41} & K_{55} & K^4_{12}+K^6_{32} & K^5_{42} & K^5_{43}+K^6_{34} & K^6_{31} \\
0 & 0 & K^4_{23} & 0 & K^4_{21}+K^6_{23} & K^4_{22}+K^6_{22} & 0 & K^6_{24} & K^6_{21} \\
0 & 0 & 0 & K^5_{21} & K^5_{24} & 0 & K^5_{22} & K^5_{23} & 0 \\
0 & 0 & 0 & K^5_{31} & K^5_{34}+K^6_{43} & K^6_{42} & K^5_{32} & K^5_{33}+K^6_{44} & K^6_{41} \\
0 & 0 & 0 & 0 & K^6_{13} & K^6_{12} & 0 & K^6_{14} & K^6_{11}
\end{pmatrix}
$$

with

$$K_{55} = K^1_{33} + K^2_{11} + K^3_{22} + K^4_{11} + K^5_{44} + K^6_{33}.$$

2D Assembly of element equations

- Let's consider a simple example to explain flux balance and BC for the assembled system $[K]\,U = F + Q$



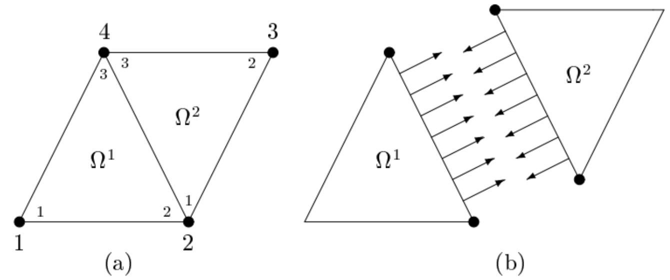
(a) (b)

$$\begin{pmatrix} K^1_{11} & K^1_{12} & 0 & K^1_{13} \\ K^1_{21} & K^1_{22}+K^2_{11} & K^2_{12} & K^1_{23}+K^2_{13} \\ 0 & K^2_{21} & K^2_{22} & K^2_{32} \\ K^1_{31} & K^1_{32}+K^2_{31} & K^2_{32} & K^1_{33}+K^2_{33} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} F^1_1 \\ F^1_2+F^2_1 \\ F^2_2 \\ F^1_3+F^2_3 \end{pmatrix} + \begin{pmatrix} Q^1_1 \\ Q^1_2+Q^2_1 \\ Q^2_2 \\ Q^1_3+Q^2_3 \end{pmatrix}$$

2D Assembly of element equations

- Here the balance must be imposed on nodes 2 and 4 remember that Q_{ij}^{k} means

the flux on node i corresponding to the contribution of edge j



Consider node 2:

$$Q_2 = Q_2^1 + Q_1^2 = (Q_{21}^1 + Q_{22}^1 + Q_{23}^1) + (Q_{11}^2 + Q_{12}^2 + Q_{13}^2) = Q_{21}^1 + Q_{23}^1 + \underbrace{(Q_{22}^1 + Q_{13}^2)}_{=0} + Q_{11}^2 + Q_{12}^2.$$

by construction we also have $Q_{23}^1 = Q_{12}^2 = 0,$ therefore

$Q_2 = Q_{21}^1 + Q_{11}^2.$, that have to be **defined on the BC** of the problem.