Mètodes Numèrics:

A First Course on Finite Elements

Plane Elasticity

Following: *Curs d'Elements Finits amb Aplicacions* (J. Masdemont)
http://hdl.handle.net/2099.3/36166

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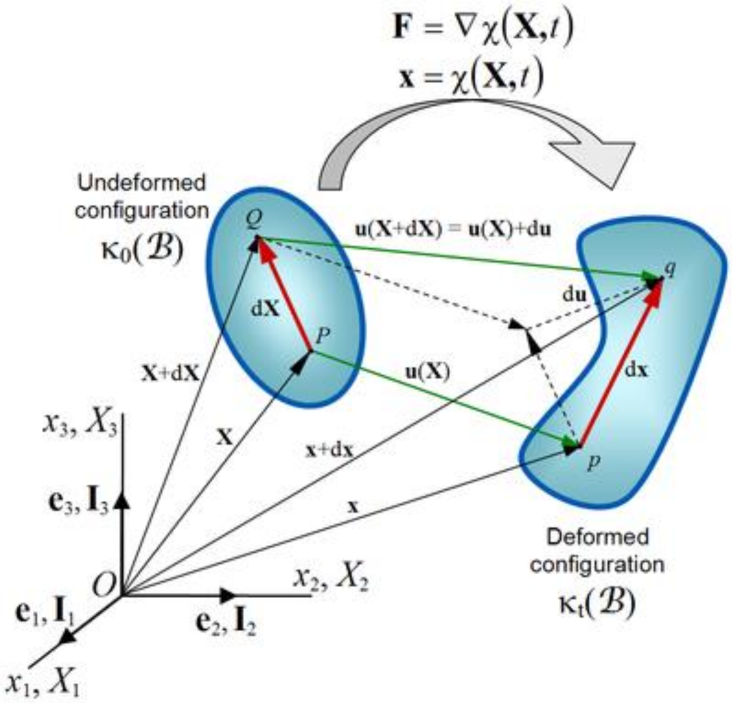
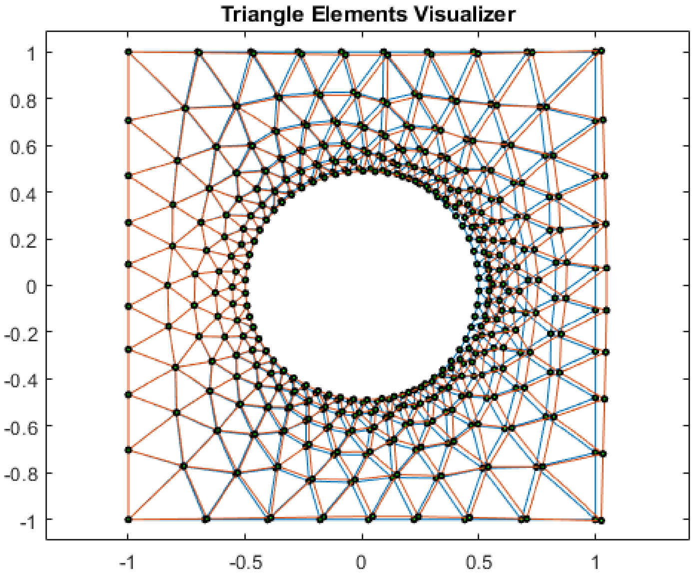
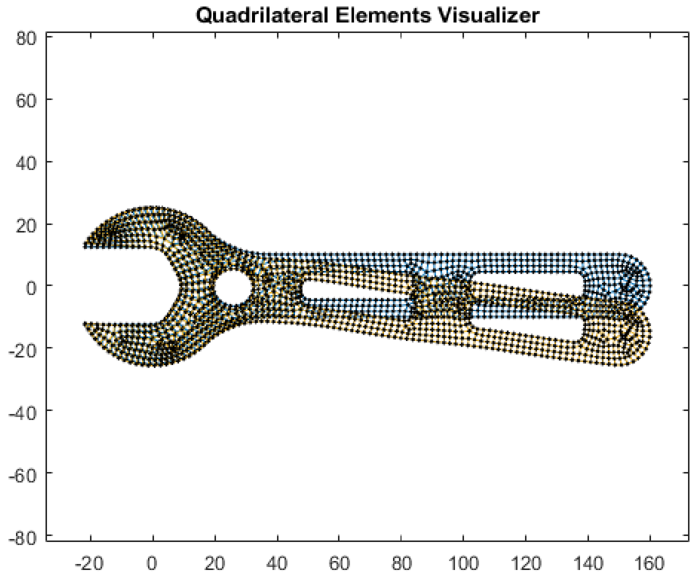
CONTINUUM MECHANICS

Continuum Mechanics



https://en.wikipedia.org/wiki/Finite_strain_theory

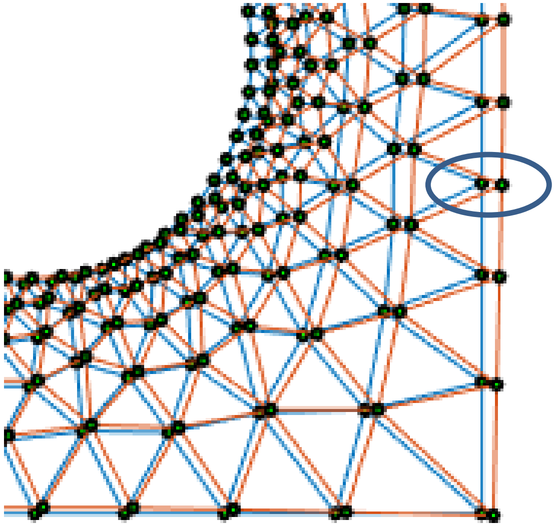




Continuum Mechanics

- Fundamental magnitudes:
  - **Displacements**: Motion of each point

$$\mathbf{u}(\mathrm{x}) = \chi(\mathrm{x}) - \mathrm{x}$$



Continuum Mechanics

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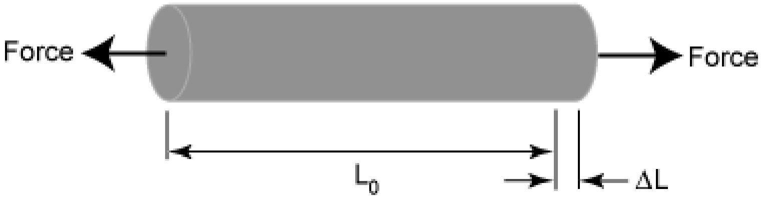
  - **Displacements**: Motion of each point

$$\mathbf{u}(\mathrm{x}) = \chi(\mathrm{x}) - \mathrm{x}$$

  - **Strain**: Relative elongation (or compression) of the material.

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathrm{x}) = \frac{\mathrm{d}\mathbf{u}}{\mathrm{dx}}$$



$$\text{Strain} = \frac{\text{Elongation}}{\text{Orininal Length}} = \frac{\Delta L}{L_0}$$

Continuum Mechanics

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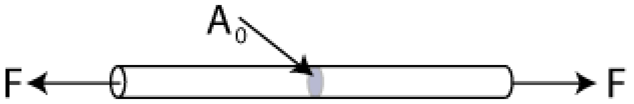
  - **Stress**: Force per unit area.

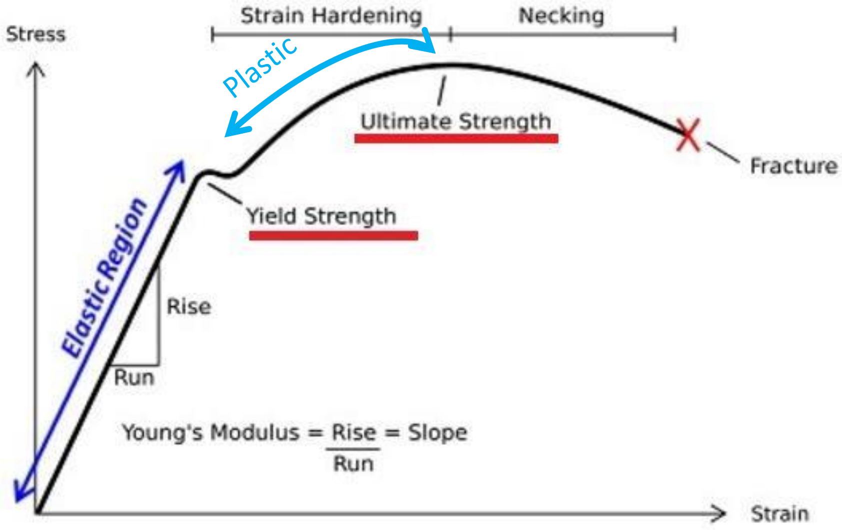
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathrm{x})$$



$$\text{Stress, } \sigma = \frac{\text{Force}}{\text{Cross-Sectional Area}} = \frac{F}{A_0}$$

Continuum Mechanics

- Strain – Stress curve: (material behavior)



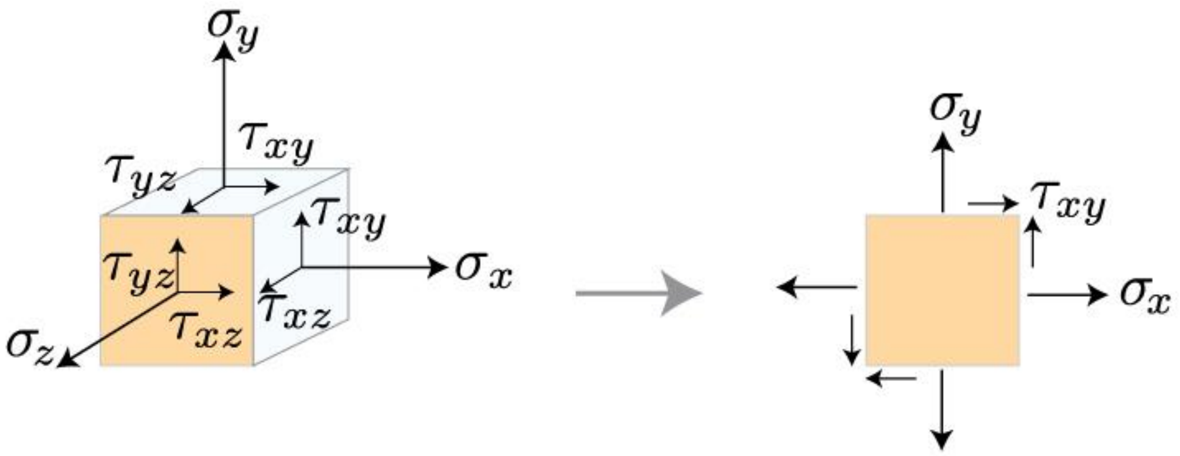
- Linear deformation:

$$\boldsymbol{\sigma}(\mathrm{x}) = \boldsymbol{C} \cdot \boldsymbol{\epsilon}(\mathrm{x})$$

Linear Matrix depending on the materials

PLANE ELASTICITY

Plane Elasticity Problems



3D Stress State

Plane Stress

$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} \longrightarrow \sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Plane Elasticity Problems

- In both cases the problem is a simplification of the 3D elasticity problem relating stress ($\boldsymbol{\sigma}$) and strain ($\boldsymbol{\epsilon}$):

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}$$

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- For the 2D case:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ 2\gamma_{xy} \end{pmatrix}$$

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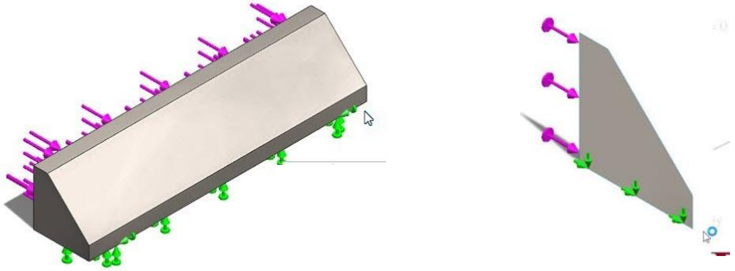
Here $\epsilon_x = \frac{\partial u}{\partial x}$, $\epsilon_y = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ being $u(x, y)$ the **displacement** function.

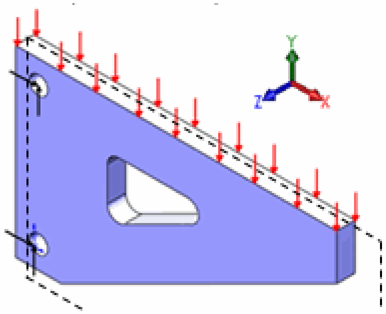
Plane Elasticity Problems

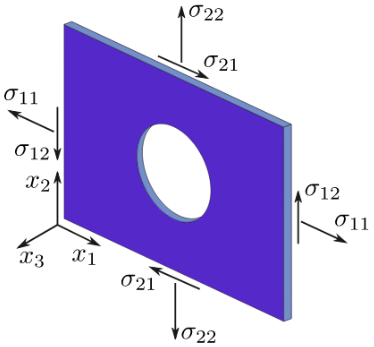
To describe the **2D elasticity problems**, we consider two main situations according to the domain geometry:

- **Plane Strain problems:** when the thickness is very large compared to the section area. It is like to study only the 2D section of a 3D domain.



- **Plane Stress problems:** when the thickness is small compared to the section area. It is like study a 2D domain with thickness.





Plane Elasticity Problems

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix}$$

E Young modulus
ν Poisson's ratio

Plane stress (t_h)

$$c_{11} = c_{22} = \frac{E}{1-\nu^2}$$

$$c_{12} = c_{21} = \nu c_{11}$$

$$c_{33} = \frac{E}{2(1+\nu)}$$

Plane strain $(t_h = 1)$

$$c_{11} = c_{22} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$c_{12} = c_{21} = \frac{\nu}{1-\nu} c_{11}$$

$$c_{33} = \frac{E}{2(1+\nu)}$$

FEM FOR PLANE ELASTICITY

FEM Plane Elasticity Problems

Degrees of freedom

For each node (x, y) we have two possible **displacements** $(u_x, u_y) \equiv (u, v)$

$$\boxed{u = u(x, y),} \qquad \boxed{v = v(x, y)}$$

FEM Plane Elasticity Problems

- Model equation system $u = u(x,y),\ v = v(x,y)$

$$\begin{cases} -\dfrac{\partial}{\partial x}\left(c_{11}\dfrac{\partial u}{\partial x} + c_{12}\dfrac{\partial v}{\partial y}\right) - \dfrac{\partial}{\partial y}\left(c_{33}\dfrac{\partial u}{\partial y} + c_{33}\dfrac{\partial v}{\partial x}\right) = F_x, \\ -\dfrac{\partial}{\partial x}\left(c_{33}\dfrac{\partial u}{\partial y} + c_{33}\dfrac{\partial v}{\partial x}\right) - \dfrac{\partial}{\partial y}\left(c_{12}\dfrac{\partial u}{\partial x} + c_{22}\dfrac{\partial v}{\partial y}\right) = F_y, \end{cases}$$

For an ***isotropic*** *material*

Plane strain

$$c_{11} = \frac{E(1-\nu)}{1-\nu-2\nu^2}, \quad c_{12} = \frac{E\,\nu}{1-\nu-2\nu^2}, \quad c_{22} = \frac{E(1-\nu)}{(1-\nu-2\nu^2)}, \quad c_{33} = G = \frac{E}{2(1+\nu)}.$$

Plane stress

$$c_{11} = c_{22} = \frac{E}{1-\nu^2}, \quad c_{12} = \frac{E\,\nu}{1-\nu^2}, \quad c_{33} = G = \frac{E}{2(1+\nu)}.$$

TRIANGLE PLANE ELASTICITY

Plane Elasticity: **Engineering Notation**

- For **triangle elements**, according to the interpolation expression

$$u = \psi_1(x,y)u_1 + \psi_2(x,y)u_2 + \psi_3(x,y)u_3$$

$$v = \psi_1(x,y)v_1 + \psi_2(x,y)v_2 + \psi_3(x,y)v_3$$

We can also write it in matrix form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \psi_1 & 0 & \psi_2 & 0 & \psi_3 & 0 \\ 0 & \psi_1 & 0 & \psi_2 & 0 & \psi_3 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \boldsymbol{\Psi u}$$

Plane Elasticity: **Engineering Notation**

Now the **strain** is expressed as

$$\boldsymbol{\epsilon} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} = \left(\underbrace{\begin{matrix} \dfrac{\partial \psi_1}{\partial x} & 0 \\ 0 & \dfrac{\partial \psi_1}{\partial y} \\ \dfrac{\partial \psi_1}{\partial y} & \dfrac{\partial \psi_1}{\partial x} \end{matrix}}_{B_1} \quad \underbrace{\begin{matrix} \dfrac{\partial \psi_2}{\partial x} & 0 \\ 0 & \dfrac{\partial \psi_2}{\partial y} \\ \dfrac{\partial \psi_2}{\partial y} & \dfrac{\partial \psi_2}{\partial x} \end{matrix}}_{B_2} \quad \underbrace{\begin{matrix} \dfrac{\partial \psi_3}{\partial x} & 0 \\ 0 & \dfrac{\partial \psi_3}{\partial y} \\ \dfrac{\partial \psi_3}{\partial y} & \dfrac{\partial \psi_3}{\partial x} \end{matrix}}_{B_3} \right) \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

Or in compact notation

$$\boldsymbol{\epsilon} = (B_1, B_2, B_3)\boldsymbol{u} = \boldsymbol{B}\boldsymbol{u}$$

and we have already the stress-strain relation

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = \boldsymbol{C}\boldsymbol{B}\boldsymbol{u}$$ Stress - displacements

Stiffness Matrix

- From the weak form we obtain ($t_h \equiv$ thickness)

$$K^e = \iint_{\Omega^e} \boldsymbol{B}^T C \boldsymbol{B}\, t_h \, d\Omega$$

$$K_{i,j}^{k,11} = a_{11} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial x}(x,y) dx dy$$

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That can be also written as:

$$K^e = \iint_{\Omega^e} \begin{pmatrix} B_1^T \\ B_2^T \\ B_3^T \end{pmatrix} C \, (B_1, B_2, B_3) t_h \, d\Omega$$

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$$K^e = \iint_{\Omega^e} \begin{pmatrix} B_1^T C B_1 & B_1^T C B_2 & B_1^T C B_3 \\ \ddots & B_2^T C B_2 & B_2^T C B_3 \\ sym & \ddots & B_3^T C B_3 \end{pmatrix} t_h\, d\Omega$$

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Or by components

$$\boxed{K_{ij}^e = \iint_{\Omega^e} B_i^T C B_j \, t_h \, d\Omega}$$

2x2 matrix

Constant-Strain Triangular elements (CST) (Linear Triangle Elements)

In the case of **triangles** $\frac{\partial \psi_i}{\partial x} = \frac{\beta_i}{2A_e}, \quad \frac{\partial \psi_i}{\partial y} = \frac{\gamma_i}{2A_e}$

$$B = \begin{pmatrix} \frac{\partial \psi_1}{\partial x} & 0 & \frac{\partial \psi_2}{\partial x} & 0 & \frac{\partial \psi_3}{\partial x} & 0 \\ 0 & \frac{\partial \psi_1}{\partial y} & 0 & \frac{\partial \psi_2}{\partial y} & 0 & \frac{\partial \psi_3}{\partial y} \\ \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_2}{\partial y} & \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_3}{\partial y} & \frac{\partial \psi_3}{\partial x} \end{pmatrix}$$

$$\boxed{B = \frac{1}{2A_e} \begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix}}$$

CST Elements

- Due to the linearity of the elements shape functions, B is a constant matrix and then

$$\boxed{K^e = t_h A_e B^T C B}$$

is also a **6x6 constant matrix**.

Constant-Strain Triangular elements (CST)
(Linear Triangle Elements)

Or using the 2x2 matrices

$$K_{ij}^e = \iint_{T^e} \frac{1}{2A_e} \begin{pmatrix} \beta_i & 0 & \gamma_i \\ 0 & \gamma_i & \beta_i \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \frac{1}{2A_e} \begin{pmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{pmatrix} t_h dT$$

$$K_e = \begin{pmatrix} K_{11}^e & K_{12}^e & K_{13}^e \\ K_{21}^e & K_{22}^e & K_{23}^e \\ K_{31}^e & K_{32}^e & K_{33}^e \end{pmatrix}$$

CST Elements

- Example: For the **reference triangular element**

$$B = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

If we consider all the elements of **C equal 1** we obtain

$$K^R = \begin{pmatrix} 1 & 1 & -1/2 & -1/2 & -1/2 & -1/2 \\ 1 & 1 & -1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 0 & 0 & 1/2 \\ -1/2 & -1/2 & 0 & 1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 0 & 1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

Element Equations

Element System

$$K^e u = F^e + Q^e$$

- **Body internal forces:** F^e

By definition

$$F^e = \iint_{\Omega^k} \mathbf{\Psi} f_b \, h d\Omega$$

where f_b means **body internal force**. Usually body forces are uniformly distributed so f_b is constant at each node.

Element Equations

Therefore f_b is a constant initial value (many times $f_b = 0$)

$$F^e = \iint_{\Omega^k} \boldsymbol{\Psi} \begin{pmatrix} f_{bx}^0 \\ f_{by}^0 \end{pmatrix} t_h d\Omega$$

For a triangular element:

$$\boxed{F^e = \frac{A_e t_h}{3} \begin{pmatrix} f_{bx}^0 \\ f_{by}^0 \\ f_{bx}^0 \\ f_{by}^0 \\ f_{bx}^0 \\ f_{by}^0 \end{pmatrix}}$$

Element Equations

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ρ density
g gravity

When **self-weight** is considered $\left(f_{bx}^0, \quad f_{by}^0\right) = \boxed{(0, \quad -\rho g)}$

Element Equations

Boundary conditions:

Essential BC: Usual values are **displacements** $u_i = 0$ or $v_i = 0$ for some of the boundary nodes.

Element Equations

Boundary conditions:

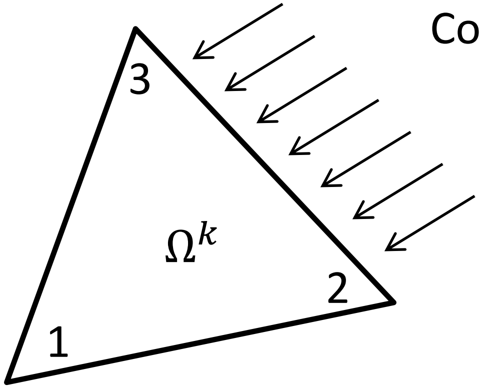
Essential BC: Usual values are **displacements** $u_i = 0$ or $v_i = 0$ for some of the boundary nodes.

Natural BC: The applied loads in a boundary element implies to compute vector Q^e. These are named **surface tractions** $\left(t_x, t_y\right)$ and their computation involves the evaluation of line integrals.

$$Q^e = \oint_{\partial\Omega^k} \mathbf{\Psi} \begin{pmatrix} t_x \\ t_y \end{pmatrix} ds$$

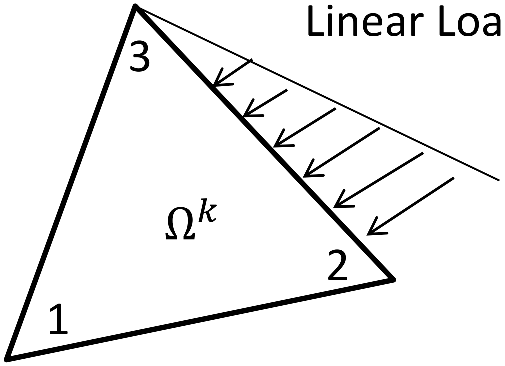
Plane Elasticity Problems: Triangles

- **Boundary Conditions**: We will consider two cases (for edge 2-3):



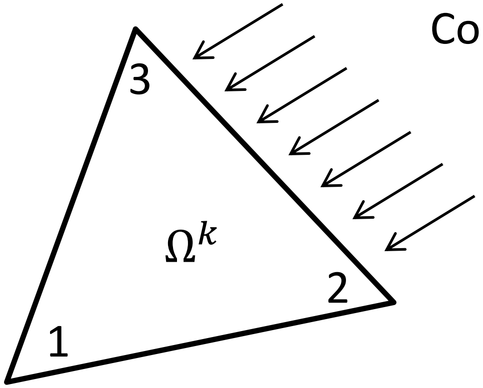
Constant Load

(negative sign against normal direction)



Linear Load

Plane Elasticity Problems: Triangles



Constant Load

Like in the FEM2D boundary conditions examples, you have to express the traction force as a 2D vector $\boldsymbol{t} = \left(t_{0x}, t_{0y}\right)$ and then these constant values are **averaged** on the two nodes involved.

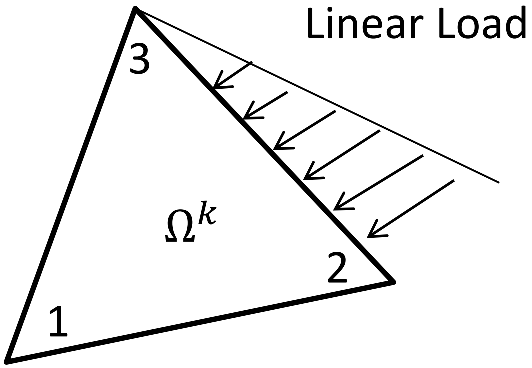
$$Q^k_{\{2-3\}} = -\frac{h^k_2}{2}(0,0,t_{0x},t_{0y},t_{0x},t_{0y})$$

Analogously

$$Q^k_{\{1-2\}} = -\frac{h^k_1}{2}(t_{0x},t_{0y},t_{0x},t_{0y},0,0)$$

$$Q^k_{\{3-1\}} = -\frac{h^k_3}{2}(t_{0x},t_{0y},0,0,t_{0x},t_{0y})$$

Plane Elasticity Problems: Triangles



Like in the FEM2D boundary conditions examples, you have to express the traction force as a 2D vector $\boldsymbol{t} = \left(t_{0x}, t_{0y}\right)$ and then these constant values are **weighted** on the two nodes involved.

$$Q^k_{\{2-3\}} = -\frac{h^k_2}{6}(0,0,2t_{0x}, 2t_{0y}, t_{0x}, t_{0y})$$

Analogously

$$Q^k_{\{1-2\}} = -\frac{h^k_1}{6}(2t_{0x}, 2t_{0y}, t_{0x}, t_{0y}, 0,0)$$

$$Q^k_{\{3-1\}} = -\frac{h^k_3}{6}(t_{0x}, t_{0y}, 0,0, 2t_{0x}, 2t_{0y})$$

Plane Elasticity Problems: Triangles

- **<u>PostProcess</u>: Strain**

Remember that strain is defined as

$$\boldsymbol{\epsilon} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} = \boldsymbol{B}\boldsymbol{u}$$

Then strain in x and y coordinates can be obtained from this product.

Plane Elasticity Problems: Triangles

- **<u>PostProcess</u>: Stress**

Remember that stress is defined as

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \boldsymbol{C\epsilon} = \boldsymbol{CBu}$$

The **principal stress** are defined as the eigenvalues of $\boldsymbol{\sigma}$**,** they are denoted as (σ_1, σ_2) respectively.

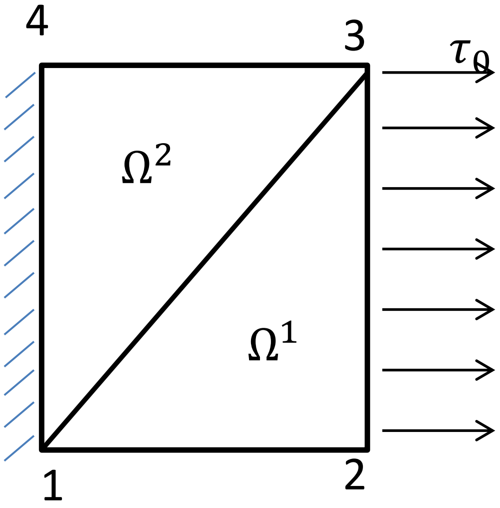
$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy} \pm \sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\tau_{xy}^2}}{2}$$

Moreover, the **Von Mises stress** value is also used to represent stress in each element

$$\boxed{\sigma_{VM} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2}}$$

Example:

Consider a rectangular piece of 120×160mm and **thickness** $t_h = 0.036$mm. It is **fixed** to the wall (left) and pulled by a constant **traction** τ_0 =1000N/mm.
Compute the **displacements** if the material properties are $E = 30 \cdot 10^6 N/mm^2$ and $\nu = 0.25$

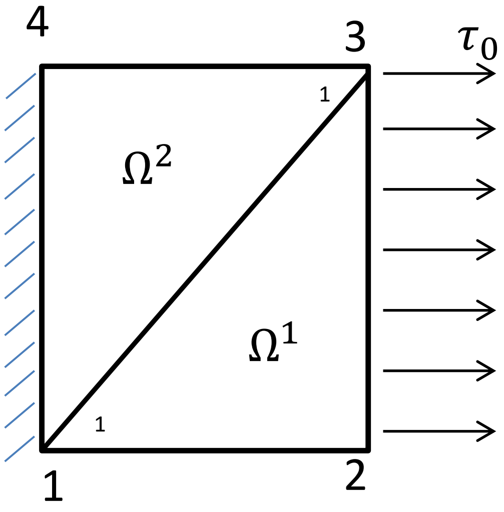


Example:

Nodes: (0,0) , (120,0), (120,160), (0,160)
Elem: [1,2,3; 3,4,1]

Plane stress

$$c_{11} = c_{22} = \frac{E}{1-\nu^2} = 32.0e+6$$
$$c_{12} = c_{21} = \nu c_{11} = 8.0e+6$$
$$c_{33} = \frac{E}{2(1+\nu)} = 12e+6$$



$$K_e = 10^5 \cdot \begin{pmatrix} 7.68 & 0.00 & -7.68 & 1.44 & 0 .00 & -1.44 \\ 0.00 & 2.88 & 2.16 & -2.88 & -2.16 & 0.00 \\ -7.68 & 2.16 & 9.30 & -3.60 & -1.62 & 1.44 \\ 1.44 & -2.88 & -3.60 & 7.20 & 2.16 & -4.32 \\ 0.00 & -2.16 & -1.62 & 2.16 & 1.62 & 0.00 \\ -1.44 & 0.00 & 1.44 & -4.32 & 0.00 & 4.32 \end{pmatrix}$$

BC:

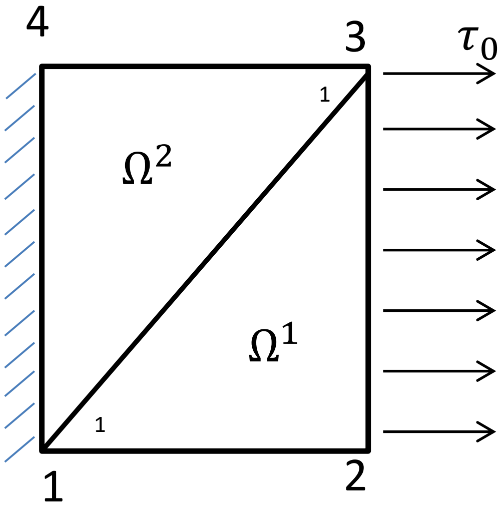
$$u_{1x} = u_{1y} = 0$$
$$u_{4x} = u_{4y} = 0$$

Loads:

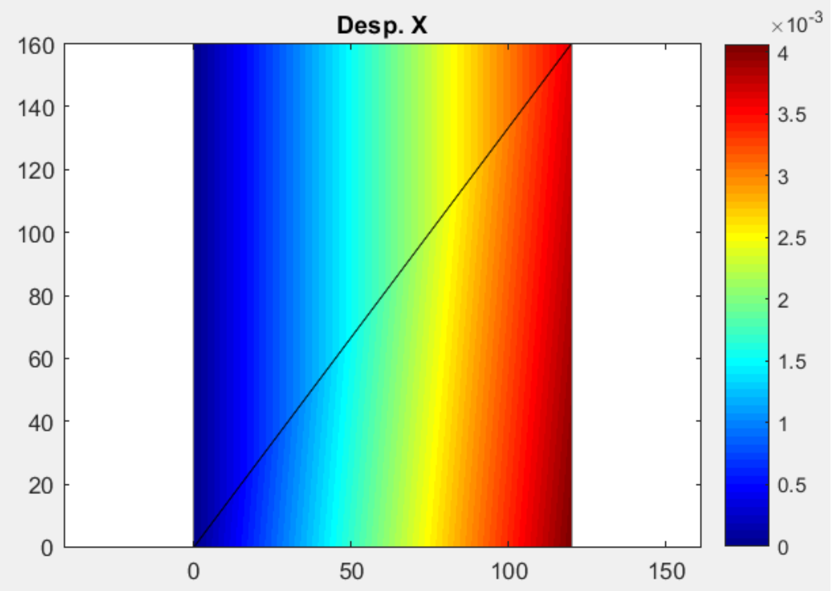
$$Q_{2x} = \frac{160}{2} \cdot 1000$$
$$Q_{3x} = \frac{160}{2} \cdot 1000$$

Example:

$$10^4\begin{pmatrix} 93 & -36 & -16.2 & 14.4 \\ -36 & 72 & 21.6 & -43.2 \\ -16.2 & 21.6 & 93 & 0 \\ 14.4 & -43.2 & 0 & 72 \end{pmatrix}\begin{pmatrix} u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{pmatrix} = 80\cdot\begin{pmatrix} 1000 \\ 0 \\ 1000 \\ 0 \end{pmatrix}$$



Solution: $u_2 = (1.1291e-01,\ \ 1.9637e-02)$
$u_3 = (1.0113e-01,\ -1.0800e-02)$



QUADRILATERAL PLANE ELASTICITY

Plane Elasticity: **Quadrilaterals**

- For **quadrilateral elements**, according to the interpolation expression

$$u = \psi_1(x,y)u_1 + \psi_2(x,y)u_2 + \psi_3(x,y)u_3 + \psi_4(x,y)u_4$$
$$v = \psi_1(x,y)v_1 + \psi_2(x,y)v_2 + \psi_3(x,y)v_3 + \psi_4(x,y)v_4$$

We have analogous terms like in the triangle case:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \psi_1 & 0 & \psi_2 & 0 & \psi_3 & 0 & \psi_4 & 0 \\ 0 & \psi_1 & 0 & \psi_2 & 0 & \psi_3 & 0 & \psi_4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}$$

Plane Elasticity: **Quadrilaterals**

- Strain

$$\boldsymbol{\epsilon} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \psi_1}{\partial x} & 0 & \dfrac{\partial \psi_2}{\partial x} & 0 & \dfrac{\partial \psi_3}{\partial x} & 0 & \dfrac{\partial \psi_4}{\partial x} & 0 \\ 0 & \dfrac{\partial \psi_1}{\partial y} & 0 & \dfrac{\partial \psi_2}{\partial y} & 0 & \dfrac{\partial \psi_3}{\partial y} & 0 & \dfrac{\partial \psi_4}{\partial y} \\ \dfrac{\partial \psi_1}{\partial y} & \dfrac{\partial \psi_1}{\partial x} & \dfrac{\partial \psi_2}{\partial y} & \dfrac{\partial \psi_2}{\partial x} & \dfrac{\partial \psi_3}{\partial y} & \dfrac{\partial \psi_3}{\partial x} & \dfrac{\partial \psi_4}{\partial y} & \dfrac{\partial \psi_4}{\partial x} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}$$

Or in compact notation

$$\boldsymbol{\epsilon} = (B_1, B_2, B_3, B_4\,)\boldsymbol{u} = \boldsymbol{B}\boldsymbol{u}$$

Plane Elasticity: **Quadrilaterals**

From the weak form we obtain

$$K^e = \iint_{\Omega^e} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} \, t_h \, d\Omega$$

Each component $K_{ij}^e = \iint_{\Omega^e} B_i^T C B_j \; t_h \, d\Omega$ is a 2x2 matrix defined as

$$K_{ij}^e = \iint_{\Omega^e} \begin{pmatrix} c_{11} \dfrac{\partial \psi_i}{\partial x} \dfrac{\partial \psi_j}{\partial x} + c_{33} \dfrac{\partial \psi_i}{\partial y} \dfrac{\partial \psi_j}{\partial y} & c_{12} \dfrac{\partial \psi_i}{\partial x} \dfrac{\partial \psi_j}{\partial y} + c_{33} \dfrac{\partial \psi_i}{\partial y} \dfrac{\partial \psi_j}{\partial x} \\ c_{33} \dfrac{\partial \psi_i}{\partial x} \dfrac{\partial \psi_j}{\partial y} + c_{21} \dfrac{\partial \psi_i}{\partial y} \dfrac{\partial \psi_j}{\partial x} & c_{33} \dfrac{\partial \psi_i}{\partial x} \dfrac{\partial \psi_j}{\partial x} + c_{22} \dfrac{\partial \psi_i}{\partial y} \dfrac{\partial \psi_j}{\partial y} \end{pmatrix} t_h \, d\Omega$$

$K_{ij}^e = K_{ji}^e$ symmetric.

Obs: The integral K_{ij}^e is computed numerically using the change of variables for derivatives into the reference element.

Plane Elasticity: **Quadrilaterals**

Once we know how to compute the stiff matrix, then we need the rest of the terms of the element equation

$$K^e u = F^e + Q^e$$

Body Forces: F^e

By definition $F^e = \iint_{\Omega^k} \mathbf{\Psi} f_b \, t_h d\Omega$ where f_b means body force. Usually body forces are uniformly distributed so f_b is constant at each node.

Plane Elasticity: **Quadrilaterals**

Therefore f_b is a constant initial value (many times $f_b = 0$)

$$F^e = \iint_{\Omega^k} \boldsymbol{\Psi} \begin{pmatrix} f_{bx}^0 \\ f_{by}^0 \end{pmatrix} t_h d\Omega$$

$$F^e = \frac{A_e t_h}{4} \begin{pmatrix} f_{bx}^0 \\ f_{by}^0 \\ f_{bx}^0 \\ f_{by}^0 \\ f_{bx}^0 \\ f_{by}^0 \\ f_{bx}^0 \\ f_{by}^0 \end{pmatrix}$$

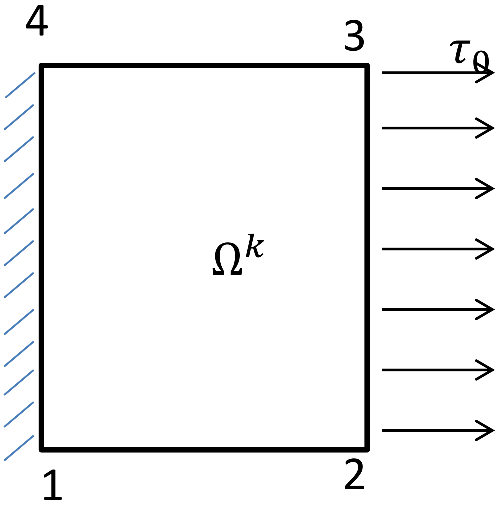
ρ density
g gravity

When **self-weight** is considered $\left(f_{bx}^0, \quad f_{by}^0\right) = (0, \quad -\rho g)$

Plane Elasticity: **Quadrilaterals**

- **Boundary Conditions**: We will consider two cases (for edge 2-3):
  - Constant τ_0

$$Q^k_{\{2-3\}} = -\frac{h_2}{2}(0,0,t_{0x},t_{0y},t_{0x},t_{0y},0,0)$$



  - Linear τ_0

$$Q^k_{\{2-3\}} = -\frac{h_2}{6}(0,0,2t_{0x},2t_{0y},t_{0x},t_{0y},0,0)$$